Phase ordering after a deep quench: Glauber dynamics for the Ising and hard-core gas on trees

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Introduction

- $G = (V, E)$ a countable infinite graph (bdd degree)
- $P_t \Leftrightarrow$ Markov semigroup of a Glauber dynamics on G with <u>multiple</u> reversible extremal Gibbs measures $\mu_1, \mu_2, \dots$.
- On finite balls the μ_i's usually selected by boundary conditions (e.g. all $+/-$ for Ising model).
- With b.c. $\Leftrightarrow \mu_i$, even if the Glauber starts from a very atypical (for μ_i) point η, $\delta_\eta P_t(\dot{}) \approx \mu_i(\cdot)$ for $t > T_{\text{mix}}$.

- **An important open problem:** is the dynamics on G able to select a specific μ_i as $t \to \infty$ if the initial configuration is sampled from a suitable distribution ν ? If so how fast ?
- Interesting case: ν highly disordered (product Bernoulli or "high-temperature" Gibbs measure).
- Finite setting formulation (G_n, P_t, μ_i): for suitable ν are the marginals of $\delta_\eta P_t$ very close to $\mu_j \neq \mu_i$ for $1 \ll t \ll T_{\text{mix}}$ and "most" η w.r.t. ν ?

- **Example**: the Ising model on $G = \mathbb{Z}^d$, $d \geq 2$. If $\beta > \beta_c$ and $\eta =$ all "$+$" $\Rightarrow \delta_\eta P_t \longrightarrow \mu^+$ while if $\eta =$ all "$-$" $\delta_\eta P_t \longrightarrow \mu^-$.
- **Conjecture** (Liggett): if $\{\eta_x\}_{x\in\mathbb{Z}^d}$ i.i.d. with $p \equiv \nu(\eta_0 = +1) > \frac{1}{2}$

$$\nu P_t \longrightarrow \mu^+ \quad \text{as } t \to \infty$$

(convergence = convergence of marginals of finite subsets of V.)

- Major obstacle: multiple Gibbs measures $\Rightarrow$ poor analysis of mixing times on finite balls.

- On $\mathbb{Z}^d$ or hexagonal lattice several results for Glauber dynamics for the Ising model at zero temperature ($\beta = \infty$) [Howard-Newman], [Newman-Stein], [Camia-De Santis-Newman], [Fontes-Schonmann-Sidoravicius].
- For our problem: $p \approx 1 \Rightarrow$ stretched exp. fixation [Fontes-Sidoravicius-Schonmann]:

$$\int d\nu(\eta)\, \mathbb{P}(\sigma_t^\eta(0) = -1) \leq e^{-t^{1/d}}$$

Related to competition between two bootstrap percolation processes.

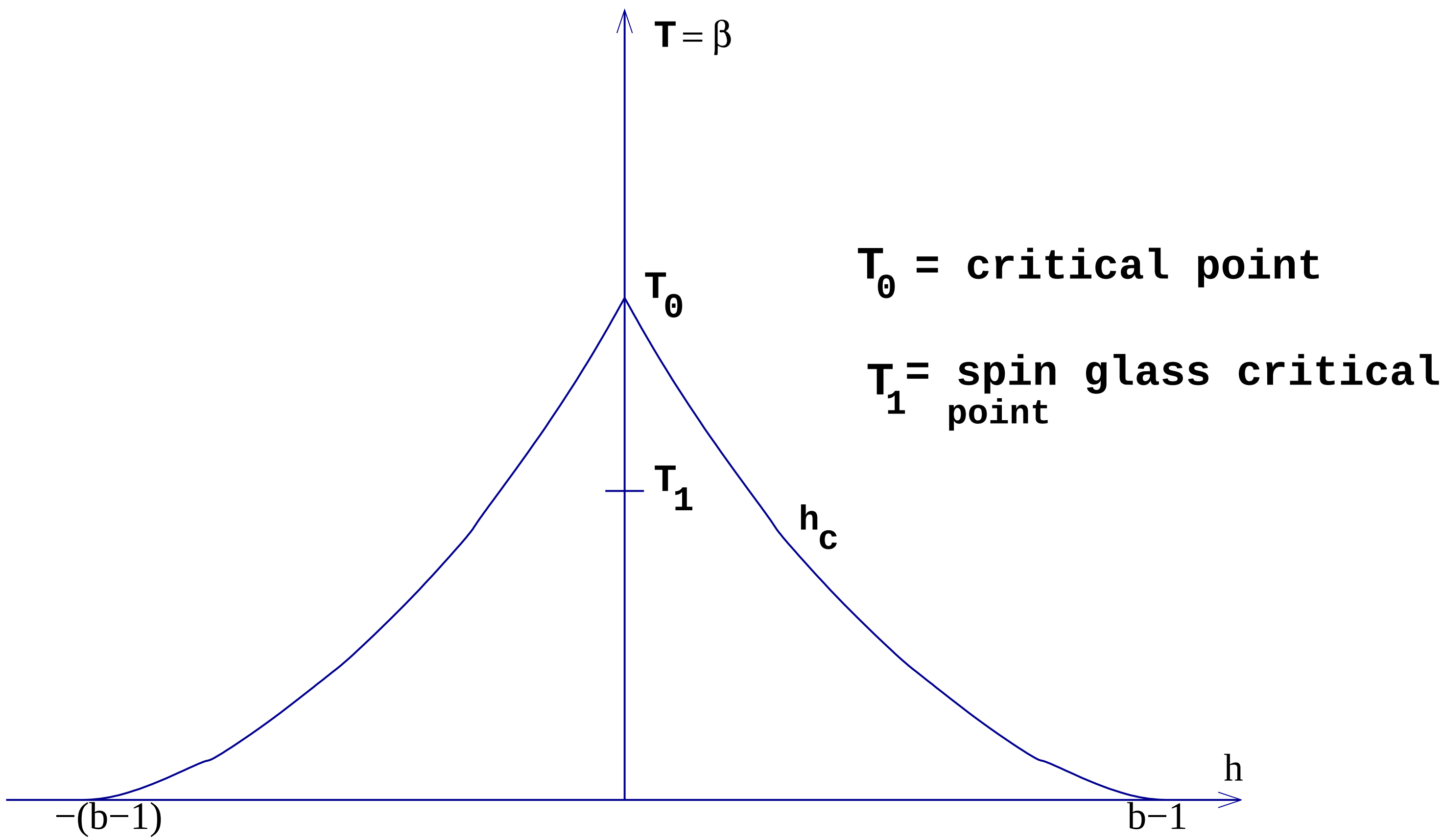
The Ising model on trees

- $T \subset \mathbb{T}^b$, the b-ary tree; E edge set of $\mathbb{T}^b$. ∂T consists of the children of its leaves.
- $\Omega = \{-1, +1\}^{T \cup \partial T}$,
  $\Omega^\tau = \{\sigma \in \Omega;\ \sigma\big|_{\partial T} = \tau\big|_{\partial T}\}$
- The Gibbs measure ($\sigma \in \Omega^\tau$):

$$\mu_T^\tau(\sigma) \propto \exp\Big[\beta\big(\sum_{x,y\in E} \sigma_x \sigma_y + h \sum_{x \in T} \sigma_x\big)\Big],$$

- $\lim_{T\uparrow\mathbb{T}^b} \mu_T^+ = \mu^+$;

The Phase Diagram



- $\mu^+ \neq \mu^-$ if $\beta > \beta_0$ and $|h| \leq h_c(\beta)$.

Glauber dynamics

On $\Omega = \Omega_T$ consider the following Markov process $t \to \sigma_t^\eta$ started from η:

- attach independent Poisson clocks of rate one, one to each vertex x;
- when the clock at x rings replace σ_x with a new value out of the conditional Gibbs measure on x conditioned on the current values of its neighbors.
- $(P_t f)(\eta) := \mathbb{E}\big(f(\sigma_t^\eta)\big)$ for local f.

Main results: preliminaries

Definition 1. Ω_α *will denote the set of starting configurations* $\eta \in \Omega$ *such that for any* $x \in \mathbb{T}^b$ *there exists a time* $t_0 = t_0(\eta, x) < \infty$ *such that for all* $t \geq t_0$

$$\left| \mathbb{E}\big(\sigma_t^\eta(x)\big) - \mu^+(\sigma(x)) \right| \leq \exp(-t^\alpha)\,.$$

- If $\eta \in \Omega_\alpha$ then $\delta_\eta P_t \to \mu^+$ weakly (convergence of marginals) as $t \to \infty$.
- Call $\mathbb{P}_p$ the product Bernoulli measure with $p := \mathbb{P}_p(\eta_x = +1)$.

Main result

Theorem 2. *In any of the following three cases* $\exists\, \alpha > 0$ *s.t.* $\nu(\Omega_\alpha) = 1$ *for any* $\nu \geq \mathbb{P}_p$*:*

i) p *large enough, any* β *but* $h > -h_c(\beta) + \epsilon(p)$
™.

ii) $p > \frac{1}{2}$, b *large enough, any* β *large enough and* $h = 0$.

iii) $p > 0$ *(even tiny),* b *large enough,* β *large,* $h = h_c(\beta)$.

(!) For β large and $h = h_c(\beta)$, starting from all -1 leads you to the minus phase.

Remarks

- The event Ω_α is increasing:
  $\eta \le \eta'$ and $\eta \in \Omega_\alpha \Rightarrow \ \eta' \in \Omega_\alpha$.

$$\mathbb{E}\big(\sigma_t^{\eta}(x)\big) \le \mathbb{E}\big(\sigma_t^{\eta'}(x)\big) \le \mathbb{E}\big(\sigma_t^{+}(x)\big)$$

  Using the fast mixing for μ^+ proved by [Sinclair-Weitz-M.] $\Rightarrow$

$$0 \le \mathbb{E}\big(\sigma_t^{+}(x)\big) - \mu^+(\sigma(x)) \le Ce^{-mt}$$

- Enough to consider $\nu = \mathbb{P}_p$.

Remarks

- Not suprising (because of fast mixing) $\mu^+(\Omega_{\alpha=1}) = 1 \Rightarrow$ any $\nu \geq \mu^+$ has $\nu(\Omega_1) = 1$.
- For any (β, h) $\mathbb{P}_p \geq \mu^+$ if $p \approx 1$ (depending on β, h).
- Hard part is to prove the result for *all* β *large enough*.
- Even if $c_{\text{sob}}(\mu^+) > 0$ (Weitz talk), methods based on decay (in time t) of relative entropy of νP_t w.r.t. μ^+ work only if $\mathrm{Ent}(\frac{d\nu}{d\mu^+}) < \infty$.

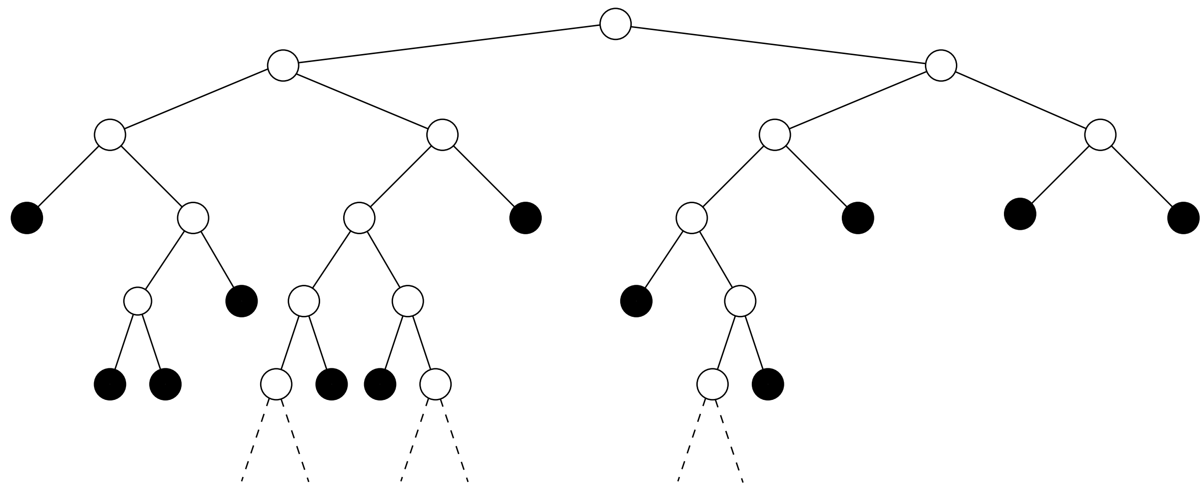
Main strategy for the proof

Need a *lower bound* on (restrict to root r):

$$\mathbb{E}\big(\sigma_t^\eta(r)\big) - \mu^+(\sigma(r))$$

- Fix a length scale $\ell = t^\alpha$ and freeze (in time) all the $\eta(x) = -1$ ℓ levels below the root;
- Call ω the configuration of "obstacles" (= frozen vertices);

$$\mathbb{E}\big(\sigma_t^\eta(r)\big) - \mu^+(\sigma_r) \geq \text{ (by monotonicity)}$$
$$\left[\mathbb{E}\big(\sigma_{t,\omega}^\eta(r)\big) - \mu_\omega^+(\sigma_r)\right] - \left[\mu^+(\sigma_r) - \mu_\omega^+(\sigma_r)\right].$$

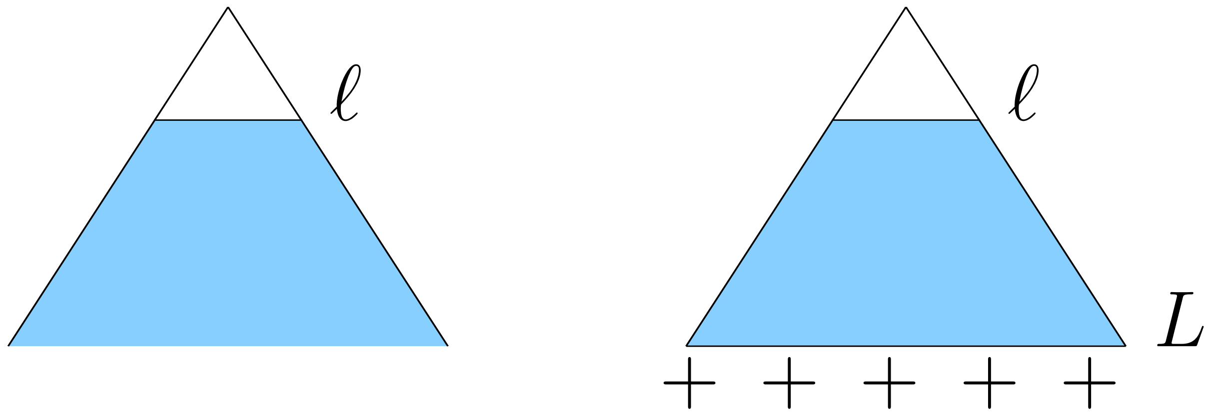


Free vertices ($\circ$) and obstacles ($\bullet$) in a given realization ω.

- Notice that $\eta = +1$ on the vertices of random tree ($\circ$ vertices) $\mathbb{T}_\omega^b$;
- Analogies with the original problem with $\eta = +1$ but on a *random* tree.

From infinite to finite tree

Fix another length scale $L = \ell^\gamma$, $\gamma > 1$;



Random obstacles below level ℓ: Infinite tree (left) and finite tree with $+$ boundary condition below level $L = \ell^\gamma$, $\gamma > 1$ (right).

- Again monotonicity $\Rightarrow \mu_\omega^{+,(L,+)} \geq \mu_\omega^+$ and $\mathbb{E}\big(\sigma_{t,\omega}^\eta(r)\big) \leq \mathbb{E}\big(\sigma_{t,\omega}^{\eta,(L,+)}(r)\big)$;

Conclusion

$$\mathbb{E}\big(\sigma_t^\eta(r)\big) - \mu^+(\sigma_r) \geq \quad \text{(previous step)}$$

$$[\mathbb{E}\big(\sigma_{t,\omega}^\eta(r)\big) - \mu_\omega^+(\sigma_r)] - [\mu^+(\sigma_r) - \mu_\omega^+(\sigma_r)]$$

$$\geq$$

$$[\mathbb{E}\big(\sigma_{t,\omega}^{\eta,(L,+)}(r) - \mu_\omega^{+,(L,+)}(\sigma_r)] \qquad \textbf{(A)}$$

$$- [\mathbb{E}\big(\sigma_{t,\omega}^{\eta,(L,+)}(r) - \mathbb{E}\big(\sigma_{t,\omega}^\eta(r)] \qquad \text{(B)}$$

$$- [\mu^+(\sigma_r) - \mu_\omega^+(\sigma_r)] \qquad \text{(C)}$$

The three steps

(A)

- $[\mathbb{E}(\sigma_{t,\omega}^{\eta,(L,+)}(r) - \mu_\omega^{+,(L,+)}(\sigma_r)]$ needs mixing times bounds for a random tree with obstacles (the frozen vertices -1);
- As in [Sinclair-Weitz-M] but now log-sob $c_{\text{sob}}(\mu_\omega^{+,(L,+)})$ shrinks as L^{-c} with large $\mathbb{P}_p$-probability;

(B) $[\mathbb{E}(\sigma_{t,\omega}^{\eta,(L,+)}(r) - \mathbb{E}(\sigma_{t,\omega}^{\eta}(r)]$ requires a true dynamical analysis (coupling);

(C) $[\mu^+(\sigma_r) - \mu_\omega^+(\sigma_r)]$ easier. Recursive analysis on random tree.

Why does it work ?

1. *Attractivity* (some monotonicity w.r.t. the partial order $\sigma \leq \eta$ iff $\sigma_x \leq \eta_x$ for all x). Attractivity (w.r.t. a different partial order) shared by the independent set model.

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3. *Rapid mixing* (exp fast) of the Glauber dynamics w.r.t μ^+.

Starting from $p = 1/2$

- What happens when $\beta > 0$, $h = 0$ and $p = \frac{1}{2}$?
- At $\beta = +\infty$ infinite (+) and (-) chains form in finite time and freeze [Howard];
- Recall the spin glass critical point $e^{-2\beta_1} = \frac{\sqrt{b}-1}{\sqrt{b}+1}$. ™

Conjecture 0. *If $\beta < \beta_1$ and $p = 1/2$ then $\nu_{1/2}$ a.s. $P_t^\eta \to \mu^o$, the Gibbs measure obtained as $\lim_{\Lambda \to \mathbb{T}^b} \mu_\Lambda$ with no boundary conditions .*

Supporting the conjecture

- Random (w.r.t. $\mathbb{P}_{p=\frac{1}{2}}$) boundary conditions $\Leftrightarrow$ free b.c.
- The random variable $P_t f(\eta) := \mathbb{E}\big(f(\sigma_t^\eta(r))\big)$ becomes a sure r.v. :

$$\mathbb{P}_{\frac{1}{2}}\Big(\eta;\ |P_t f(\eta) - \mathbb{E}_{\frac{1}{2}}(P_t f)| \geq e^{-ct}\Big) \leq e^{-c_f e^{ct}}\ .$$

  Notice that for odd (w.r.t. $\sigma \to -\sigma$) f's $\mathbb{E}_{\frac{1}{2}}\big(P_t f\big) = \mu^0(f) = 0$.

- For $\beta < \beta_1$ the free measure μ^0 is extremal;

Proof of concentration

- Proof via standard concentration bounds for $\mathbb{P}_{\frac{1}{2}}$.

$$\mathbb{P}_{\frac{1}{2}}\left(\eta;\ |F(\eta)| \geq r\right) \leq e^{-\frac{r^2}{2}}$$

for any mean zero function F with unitary Lipshitz norm

$$\|F\|_{\text{Lip}}^2 := \sum_{x\in\mathbb{T}^b} \|F(\eta^x) - F(\eta)\|_\infty^2 .$$

- Need bounds on Lipshitz norm of $P_t f$.

- Coupling + weighted Hamming distance technique of [Berger,Kenyon,Mossel,Peres] $+$ [Peres-Winkler] $\Rightarrow$

$$\|P_t f(\eta^x) - P_t f(\eta)\|_\infty \le C_f \; e^{-ct} \lambda^{d(x)}$$

$$b\lambda^2 < 1 \text{ if } \beta < \beta_1 \;\Rightarrow\; \|P_t f\|_{\mathrm{Lip}} \le C'_f e^{-ct}$$

- Concentration leads to stability of $\mathbb{E}_{\frac{1}{2}}\big(P_t f\big)$ w.r.t. perturbations of $\mathbb{P}_{\frac{1}{2}}$ with moderate exponential growth as $\ell \to \infty$ of relative entropy between marginals on first ℓ levels.