PERCOLATION OF BINARY SEQUENCES

Harry KESTEN (Cornell), V.S. (IMPA)

I. Benjamini (Weizmann),
M.E. Vares (CBPF),
Y. Zhang (Colorado)
Graph $G$: $\mathbb{Z}^d$, $\mathcal{T}$, $\mathbb{Z}_c^2$

$\mathcal{V}$ - vertex set of $G$,

$P_p$ the product measure on $\Omega = \{0, 1\}^\mathcal{V}$

$$P_p(\omega_v = 1) = p$$

\[ \Xi = \{0, 1\}^\mathbb{N} \]

A generic element $\xi = \{\xi_1, \xi_2, \ldots\}$ is called word.

**Definition.** Word $\xi$ is seen from the vertex $v$ in the configuration $\omega$ if there exists self-avoiding path $(v, v_1, v_2, \ldots)$ on $G$, starting from $v$, such that

$$\omega_{v_i} = \xi_i, \ i \geq 1.$$
$S(v) = S(v, \omega) =$collection of words which are seen from $v$ in configuration $\omega$

$$S_\infty = \bigcup_{v \in V} S(v, \omega).$$

**Theorem.** (Benjamini, Kesten) Let $\mathcal{G} = \mathbb{Z}_+^d$, and $p = 1/2$. Then for $d \geq 10$,

$$P\{S_\infty = \Xi\} = 1,$$

and for $d \geq 40$

$$P\{S(v) = \Xi \text{ for some } v\} = 1.$$
\(\mu\) - the product measure on \(\Xi\): \(\mu\{\xi = 1\} = \mu\).

For each \(\xi \in \Xi\)

\[
\rho(\xi) = P\{\xi \text{ is seen from some } v\} = 0 \text{ or } 1.
\]

and

\[
\mu\{\xi : \rho(\xi) = 1\} = 0 \text{ or } 1.
\]

(in case it is 1, we say almost all words percolate)

**Theorem.** (Kesten, Zhang, S.) Almost all words are seen on triangular lattice at \(p = 1/2\).

**Theorem.** (Kesten, Zhang, S.) All words are seen on \(\mathbb{Z}_{cp}^2\) lattice for \(p_c(\mathbb{Z}^2_{cp}) < p < 1 - p_c(\mathbb{Z}^2_{cp})\).

**Open problems:** \(p = 1/2\) and \(\mathcal{G}\) is \(\mathbb{Z}^2, \mathbb{Z}^3\).
What happens if \( p \) is close to 0 or 1 (is sub- or super-critical)?

**Theorem.** (Kesten, Vares, S.) Almost all words are seen on \( \mathbb{Z}_+^2 \) for \( p > p_c(\mathbb{Z}_+^2, \text{oriented}) \), if \( \mu \) is sufficiently close to 1.
We consider the following north-east oriented site percolation model on \( \mathbb{Z}^2_+ \): the lines

\[
H_i := \{(x, y): x + y = i\}
\]

are first declared to be “bad” (or “good”), independently of each other, with probability \( \delta \) (\( 1 - \delta \), respectively);
vertices on good lines are open with probability

\[
p_G > p_c(\mathbb{Z}^2 \text{ oriented}),
\]

vertices on bad lines they are open with probability

\[
p_B < p_c(\mathbb{Z}^2 \text{ oriented}),
\]

independently of each other, given the configuration of lines.

\[
\Theta(p_G, p_B, \delta) = P(C_0 \text{ is infinite }),
\]

where \( C_0 \) denotes the oriented open cluster of the origin.
**Theorem** (Kesten, Vares, S.) If \( p_G > p_c \) and \( p_B > 0 \) there exists \( \delta_0 = \delta_0(p_G, p_B) > 0 \) so that

\[
\Theta(p_G, p_B, \delta) > 0
\]

for all \( \delta \leq \delta_0 \).


B. McCoy, T. Wu (1968-9) Ising model with random couplings.
J. Jonasson, E. Mossel, Y. Peres (2000) Percolation on stretched lattice \( d \geq 3 \).
B. N. B. Lima (2003) \( d = 2 \). Stretched lattice, McCoy-Wu model.
C. Hoffman (2004) Stretched lattice \( d = 2 \).