Fitting and Simulation of Models for Telecommunication Access Networks

Volker Schmidt
schmidt@mathematik.uni-ulm.de

University of Ulm, Department of Stochastics

Joint work with F. Fleischer, C. Gloaguen and H. Schmidt

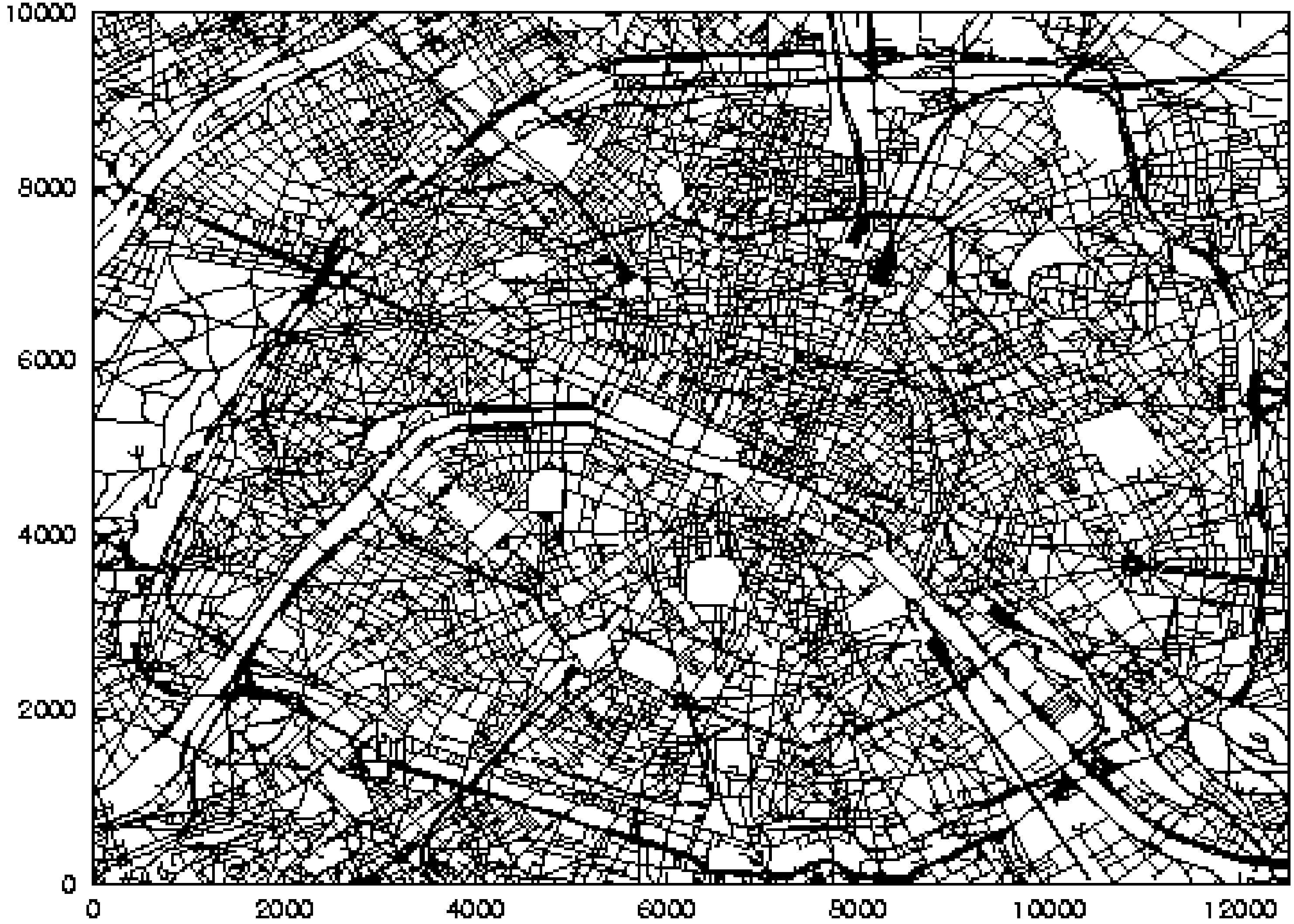
Outline

- Stochastic–geometric network modelling
  - Aims
  - The Stochastic Subscriber Line Model
- Models for the road system
  - Random tessellations
  - Nestings of tessellations
- Model choice procedure
  - Model choice based on distance measures
  - Results for real input data
- Typical Cox-Voronoi cells
  - Simulation algorithm
  - Results

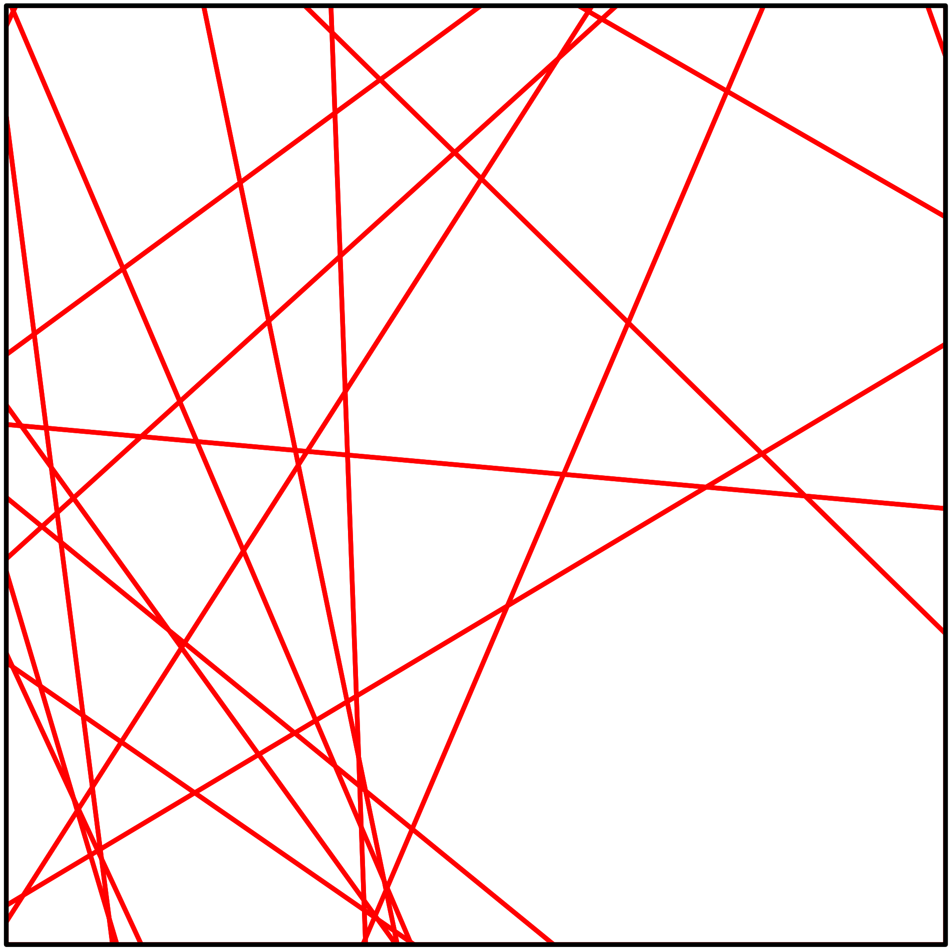
Stochastic–geometric network modelling
Aims

- Aims of modelling
  - Cost analysis and risk evaluation
  - Analysis of performance indicators
  - Simulation of present and future network design scenarios
  - Description of the network by a minimum number of structural parameters
- Models are necessary both for the road system and for the telecommunication equipment
- The model choice has to be based on the statistical analysis of real network data as well as of simulated network data

Stochastic–geometric network modelling
Infrastructure system of Paris

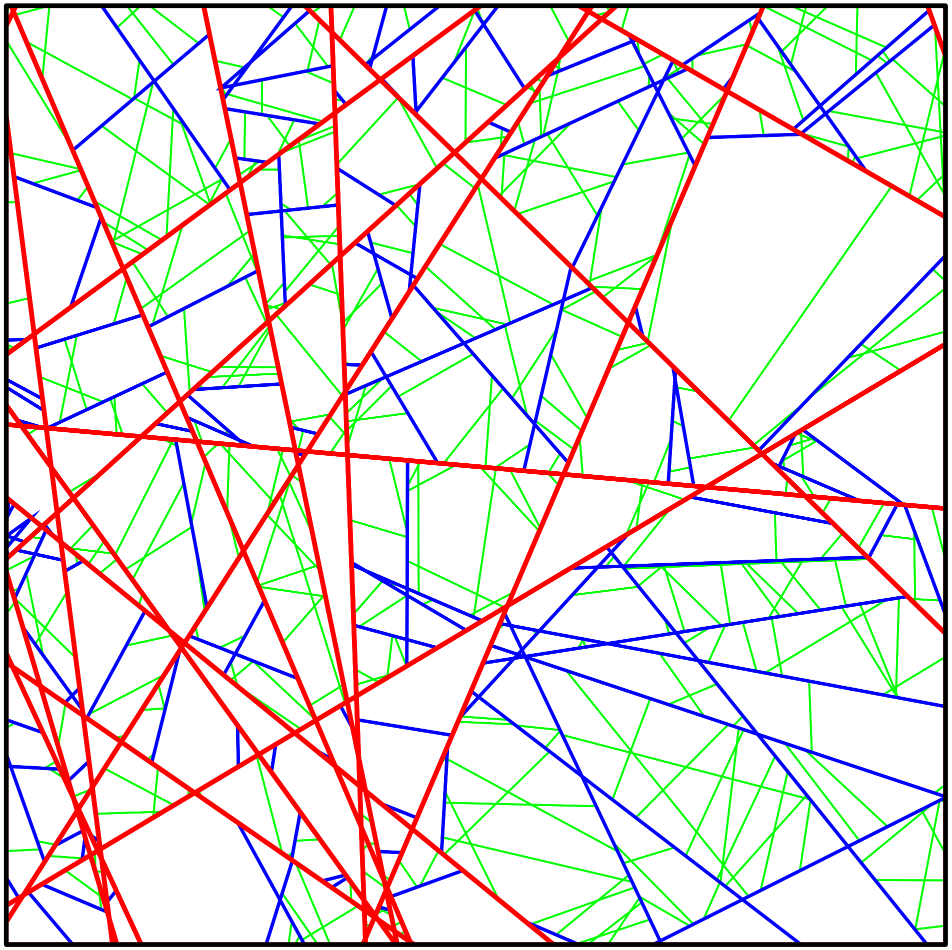


Stochastic–geometric network modelling
Stochastic Subscriber Line Model (SSLM)



System of main roads

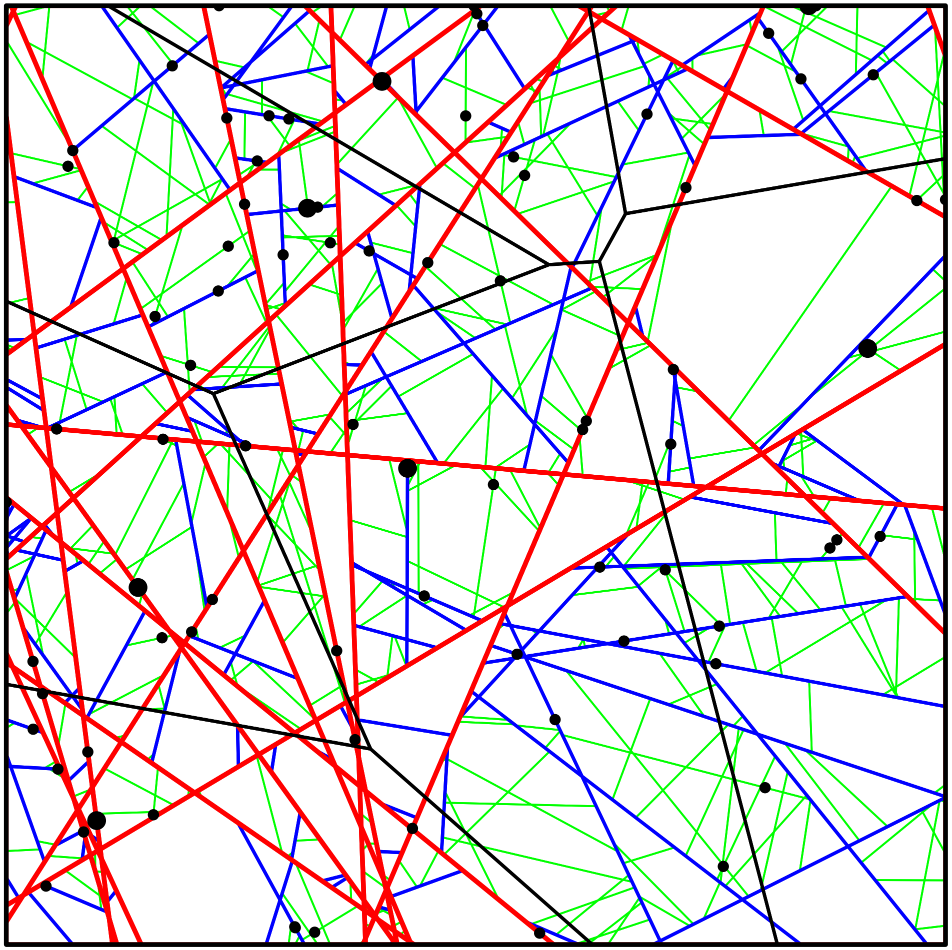
Stochastic–geometric network modelling
Stochastic Subscriber Line Model (SSLM)



System of main roads and side streets

Stochastic–geometric network modelling

Stochastic Subscriber Line Model (SSLM)



Placement of network nodes and their serving zones

Models for the road system
Random tessellations

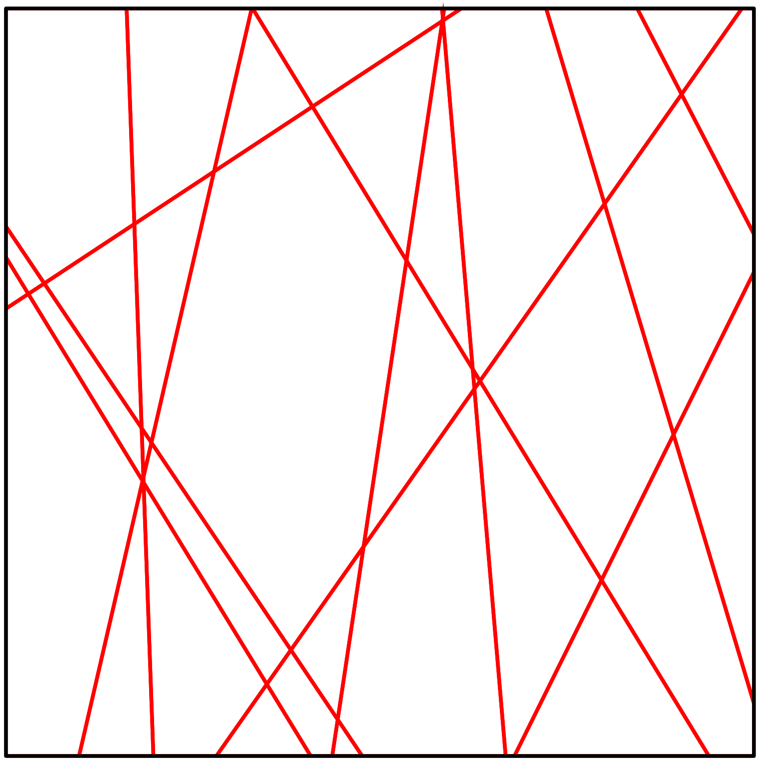
- A sequence $\{P_n\}_{n\geq 1}$ of convex polytopes $P_n \in \mathbb{R}^2$ is called (deterministic) tessellation of $\mathbb{R}^2$ if
  - $\text{int}\, P_n \neq \emptyset$ for all $n \geq 1$
  - $\text{int}\, P_n \cap \text{int}\, P_m = \emptyset$ for all $n \neq m$
  - $\bigcup_{n=1}^{\infty} P_n = \mathbb{R}^2$
  - $\sum_{n\geq 1} \mathbb{1}_{\{P_n \cap K \neq \emptyset\}} < \infty$ for all compact sets $K \in \mathbb{R}^2$
- The P_n's are called cells of the tessellation
- A sequence $X_0 = \{\Xi_n\}_{n\geq 1}$ of random convex polytopes is called random tessellation of $\mathbb{R}^2$ if

$$\mathbb{P}(X_0 \in \mathcal{T}) = 1 \,,$$

where $\mathcal{T}$ denotes the family of all tessellations in $\mathbb{R}^2$

Models for the road system

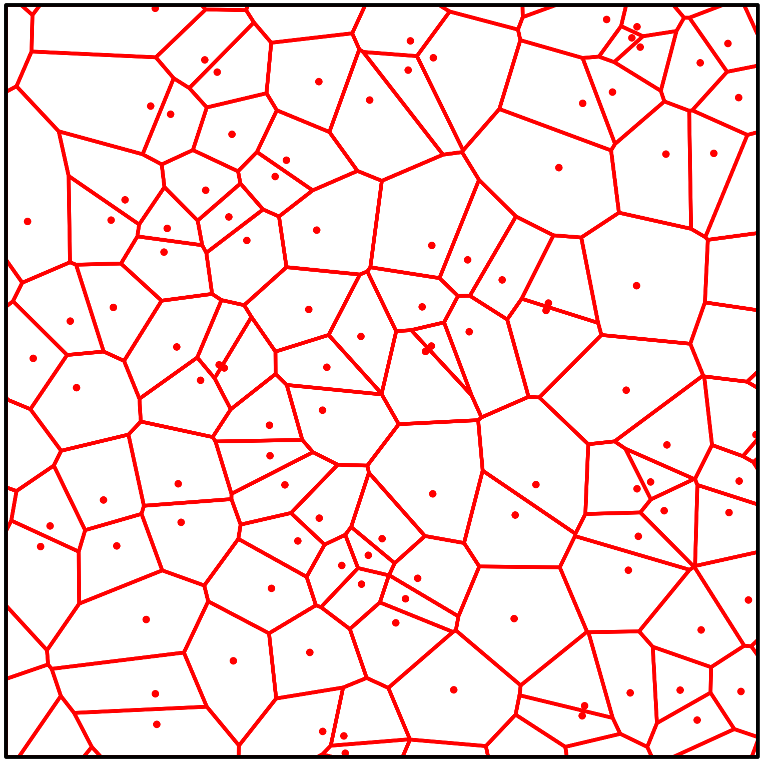
Poisson line tessellation (PLT)



Realization of PLT

- Induced by Poisson line process
- The intensity γ_{PLT} is the mean total length of lines per unit area
- Notice: $2\gamma_{PLT}$ is the mean number of lines intersecting the unit ball

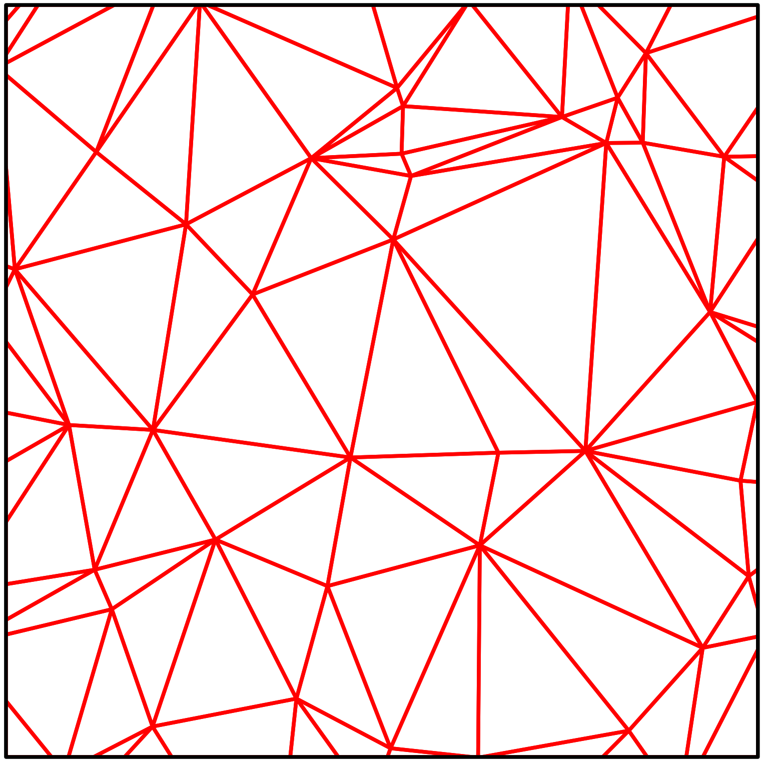
Models for the road system
Poisson–Voronoi tessellation (PVT)



Realization of PVT

- Cells are formed with respect to a set of nuclei
- The intensity γ_{PVT} is the mean number of nuclei per unit area

Models for the road system
Poisson–Delaunay tessellation (PDT)

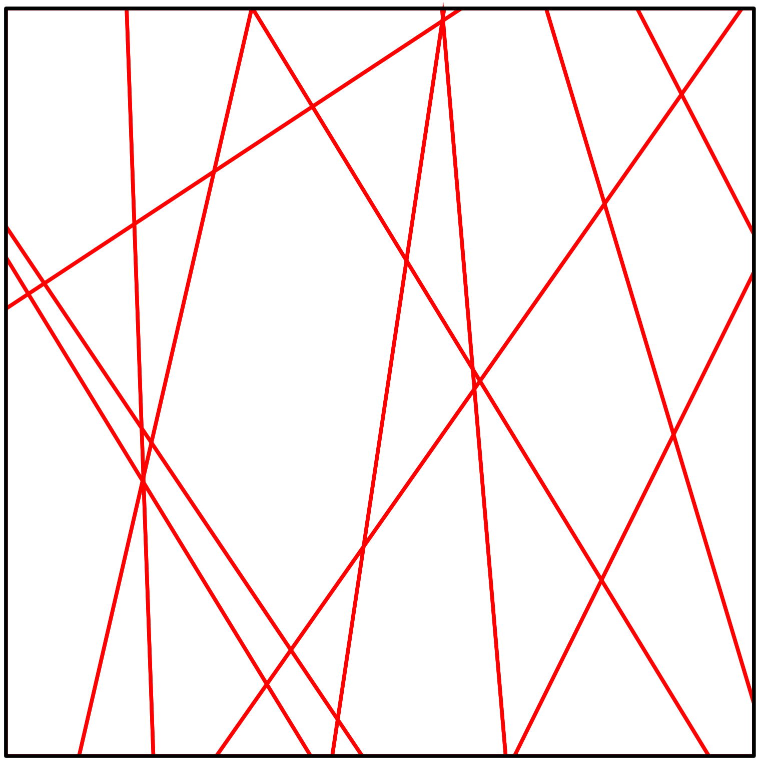


Realization of PDT

- Cells are triangles
- Vertices are nuclei of a PVT
- The intensity γ_{PDT} is the mean number of vertices per unit area

Models for the road system

Some characteristics of random tessellations



Realization of PLT

- Global characteristics
  - λ_1 mean number of vertices
  - λ_2 mean number of edges
  - λ_3 mean number of cells
  - λ_4 mean total length of edges
- Local characteristics
  - mean perimeter
  - mean area

Tessellation	λ_1	λ_2	λ_3	λ_4
PLT	$\frac{1}{\pi}\gamma^2$	$\frac{2}{\pi}\gamma^2$	$\frac{1}{\pi}\gamma^2$	γ
PVT	2γ	3γ	γ	$2\sqrt{\gamma}$
PDT	γ	3γ	2γ	$\frac{32}{3\pi}\sqrt{\gamma}$

Values of $\lambda_1, \ldots, \lambda_4$ for a tessellation with intensity γ

Models for the road system
Random nestings of tessellations

- A (deterministic) iterated tessellations of $\mathbb{R}^2$ is given by

$$\{P_n \cap P_{n\nu} : \text{int } P_n \cap \text{int } P_{n\nu} \neq \emptyset\,;\ n, \nu \in \mathbb{N}\}$$

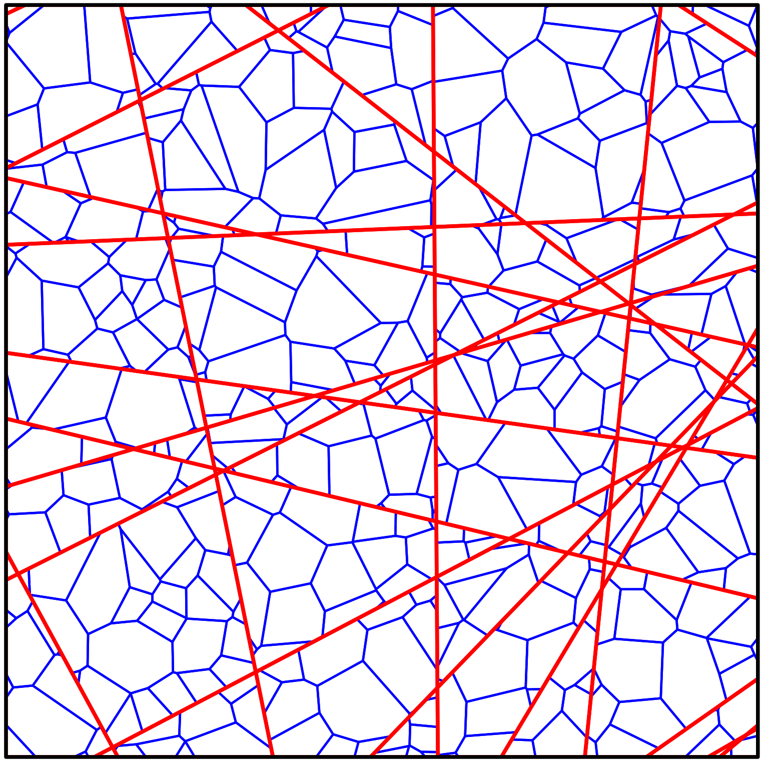
  - an initial tessellation $\{P_n\}_{n \geq 1}$
  - a sequence $(\{P_{n\nu}\}_{\nu \geq 1})_{n \geq 1}$ of component tessellations

- A (random) nesting of tessellations in $\mathbb{R}^2$ is given by

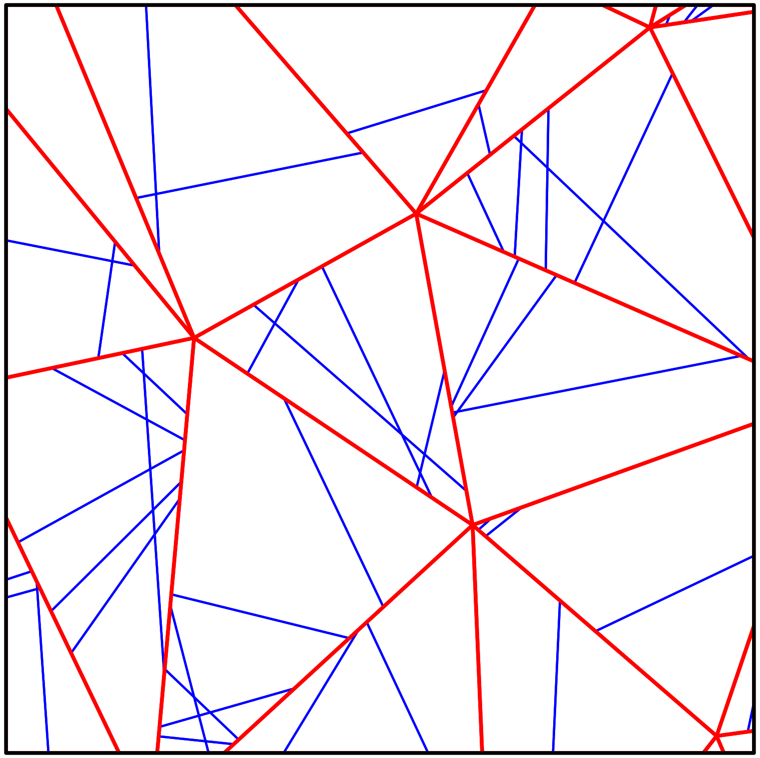
$$\{\Xi_n \cap \Xi_{n\nu} : \text{int } \Xi_n \cap \text{int } \Xi_{n\nu} \neq \emptyset\,;\ n, \nu \in \mathbb{N}\}$$

  - $X_0 = \{\Xi_n\}_{n \geq 1}$ is an arbitrary random tessellation in $\mathbb{R}^2$
  - $\{X_n\}_{n \geq 1} = (\{\Xi_{n\nu}\}_{\nu \geq 1})_{n \geq 1}$ is an independent sequence of independent and identically distributed random tessellations in $\mathbb{R}^2$
  - Notation: X_0/X_1–nesting

Models for the road system
Random nestings of tessellations

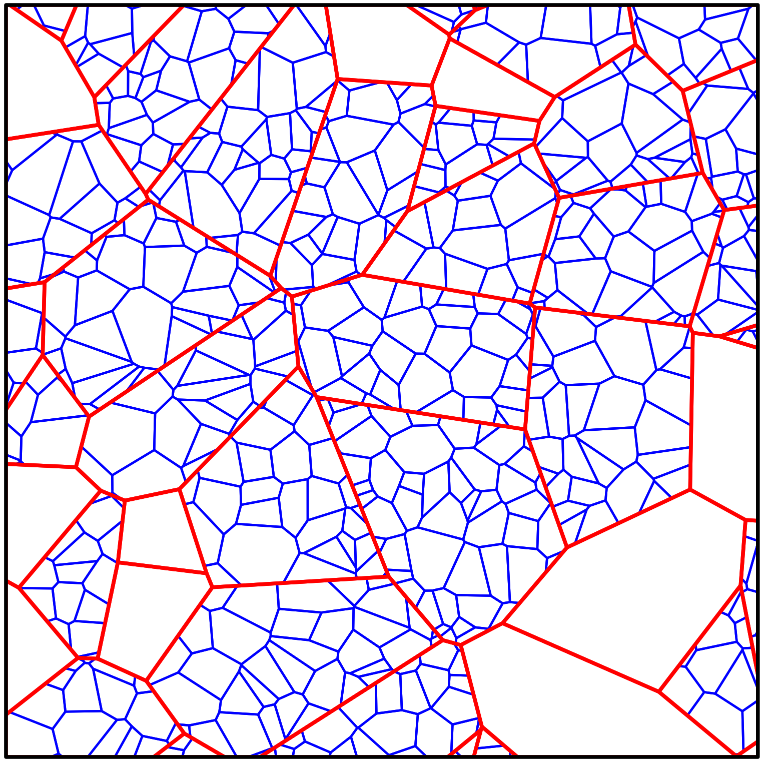


PLT/PVT–nesting

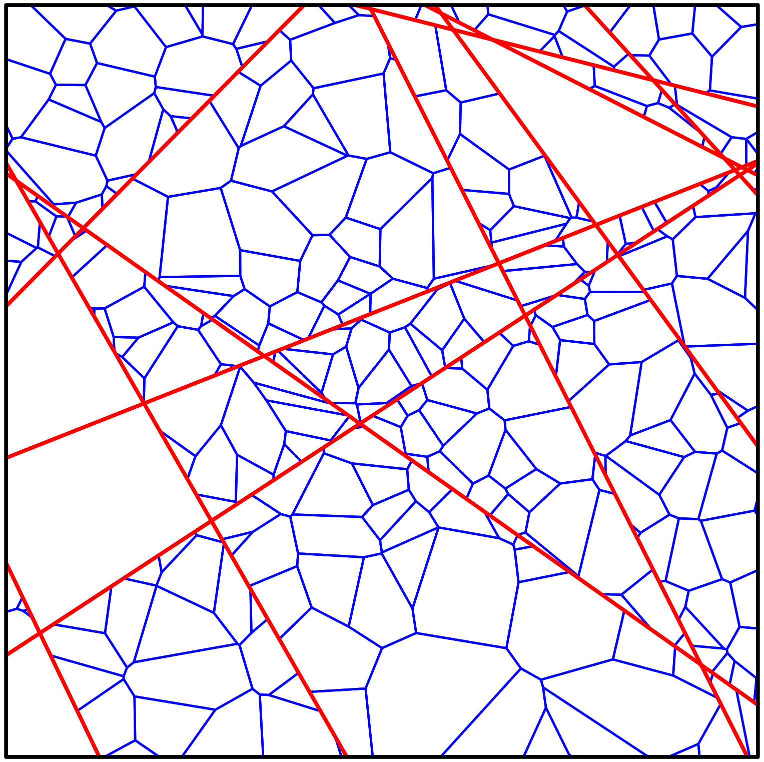


PDT/PLT–nesting

Models for the road system
Random nestings of tessellations



PVT/PVT–nesting

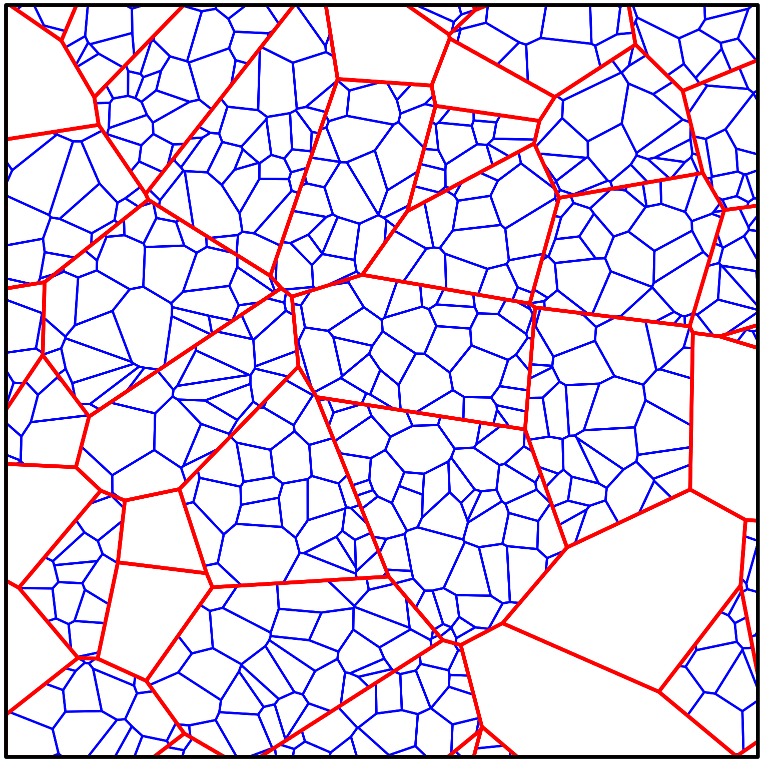


PLT/PVT-nesting

Nestings with Bernoulli thinning ($p = 0.75$)

Models for the road system

Some characteristics of random nestings



PVT/PVT-nesting

- Characteristics of X_0 $(\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}, \lambda_4^{(0)})$
- Characteristics of X_1 $(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}, \lambda_4^{(1)})$
- Measured per unit area

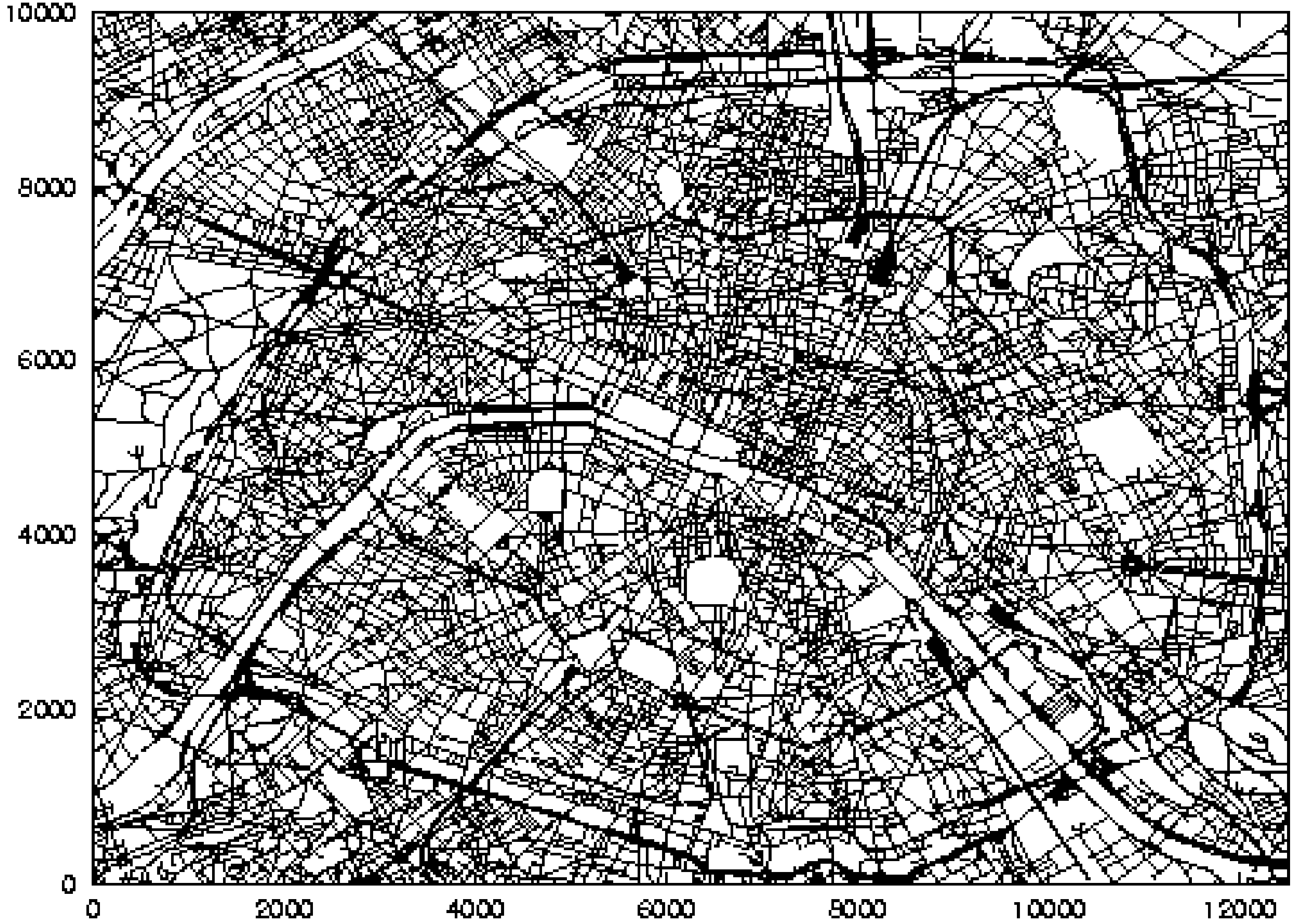
Joint characteristics of X_0/pX_1–nesting

$$\begin{aligned}
\lambda_1 &= \lambda_1^{(0)} + p\lambda_1^{(1)} + \frac{4p}{\pi}\lambda_4^{(0)}\lambda_4^{(1)} \\
\lambda_2 &= \lambda_2^{(0)} + p\lambda_2^{(1)} + \frac{6p}{\pi}\lambda_4^{(0)}\lambda_4^{(1)} \\
\lambda_3 &= \lambda_3^{(0)} + p\lambda_3^{(1)} + \frac{2p}{\pi}\lambda_4^{(0)}\lambda_4^{(1)} \\
\lambda_4 &= \lambda_4^{(0)} + p\lambda_4^{(1)}
\end{aligned}$$

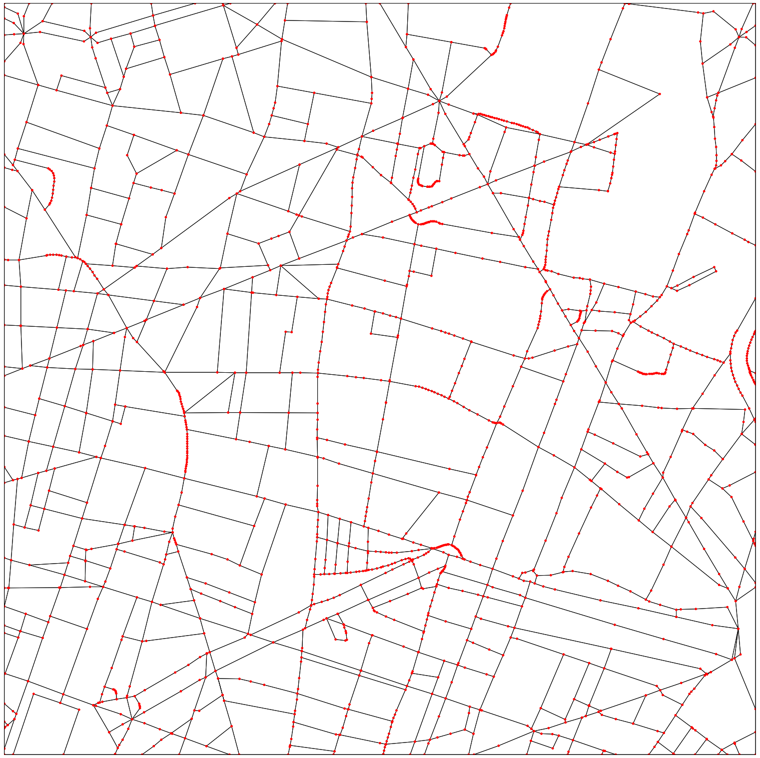
$\Rightarrow$ Formulae for nestings involving PLT, PDT and PVT

Idea of the model choice

Real network data



Idea of the model choice
Real network data



- Data possess certain hierarchical structure
- Examples of data characteristics
  - Number of road intersections (λ_1)
  - Number of edges (λ_2)
  - Number of cells (λ_3)
  - Total length of roads (λ_4)

Real network data and their characteristics

Idea of the model choice

Summary

- Observe input data in sampling window W
- Get vector of estimates

$$\boldsymbol{\lambda}^{inp} = (\lambda_1^{inp}, \lambda_2^{inp}, \lambda_3^{inp}, \lambda_4^{inp})$$

  from input data by using unbiased estimators
- For certain values of γ (or γ_0 and γ_1) compute the entries $\lambda_1, \lambda_2, \lambda_3$, and λ_4 of $\boldsymbol{\lambda}$
- Minimize distance $d(\boldsymbol{\lambda}^{inp}, \boldsymbol{\lambda}) \Rightarrow d_{min}(\boldsymbol{\lambda}^{inp}, \boldsymbol{\lambda})$
- Repeat for all competing models: $\boldsymbol{\lambda}^{opt}$ and γ^{opt}
- Validation by Monte-Carlo tests

Model choice procedure
Relative distance measures

- Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ denote two vectors
- Euclidean distance

$$d_e(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} \left(\frac{x_i - y_i}{x_i} \right)^2}$$

- Absolute value distance

$$d_a(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \left| \frac{x_i - y_i}{x_i} \right|$$

- Maximum norm distance

$$d_m(\mathbf{x}, \mathbf{y}) = \max_{i=1,\ldots,n} \left| \frac{x_i - y_i}{x_i} \right|$$

Model choice procedure
Optimization of distances

- Estimate $\boldsymbol{\lambda}^{inp} = (\lambda_1^{inp}, \lambda_2^{inp}, \lambda_3^{inp}, \lambda_4^{inp})$ from input data
- Consider PLT (γ_{PLT}), PVT (γ_{PVT}), and PDT (γ_{PDT})
  - Go through a range of values for $\gamma_{PLT}, \gamma_{PVT}$, and γ_{PDT}
  - Each time compute values $\lambda_1, \lambda_2, \lambda_3$, and λ_4 of $\boldsymbol{\lambda}$
  - Minimize distance $d = d(\boldsymbol{\lambda}^{inp}, \boldsymbol{\lambda})$
- Obtain $d_{PLT}^{min}, d_{PVT}^{min}$, and d_{PDT}^{min}
- Obtain γ^{opt} through $d^{opt} = \min\{d_{PLT}^{min}, d_{PVT}^{min}, d_{PDT}^{min}\}$
- Analogously in case of nestings

Numerical examples
Simulated input data

- Consider PLT with $\gamma = 0.1$
- Quadratic sampling window (side length 10000)

	Estimated	Theoretical
λ_1	0.00318	0.00318
λ_2	0.00630	0.00637
λ_3	0.00318	0.00318
λ_4	0.09995	0.10000

Estimated and theoretical characteristics of the PLT

Numerical examples
Simulated input data

Let $\gamma \in [0.0001, 0.5]$, step width 10^{-5}

	$d_{e,min}$	γ	$d_{a,min}$	γ	$d_{m,min}$	γ
PLT	0.0075	0.0998	0.0102	0.0999	0.0049	0.0997
PVT	0.4672	0.0020	0.7450	0.0021	0.3349	0.0021
PDT	0.6455	0.0017	1.0968	0.0016	0.4372	0.0018

Estimated and theoretical characteristics of the PLT

Numerical examples
Simulated input data

- $X_0 =$ PLT ($\gamma_0 = 0.08$), $X_1 =$ PDT ($\gamma_1 = 0.0008$)
- Quadratic sampling window (side length 10000)

	Estimated	Theoretical
λ_1	0.01231	0.012619
λ_2	0.02165	0.021147
λ_3	0.00829	0.008528
λ_4	0.17433	0.176034

Estimated and theoretical characteristics of the PLT/PDT–nesting

Numerical examples
Simulated input data

Let $\gamma_0, \gamma_1 \in [0.00001, 0.15]$, step width 10^{-5}

	$d_{e,min}$	γ_0	γ_1
PLT/PLT	0.07422	0.05727	0.10703
PLT/PVT	0.06844	0.11961	0.00052
PLT/PDT	0.04008	0.08235	0.00073
PVT/PLT	0.06844	0.00052	0.11961
PVT/PVT	0.21489	0.00001	0.00641
PVT/PDT	0.06424	0.00101	0.00119
PDT/PLT	0.04008	0.00073	0.08235
PDT/PVT	0.06424	0.00119	0.00101
PDT/PDT	0.07416	0.00074	0.00074

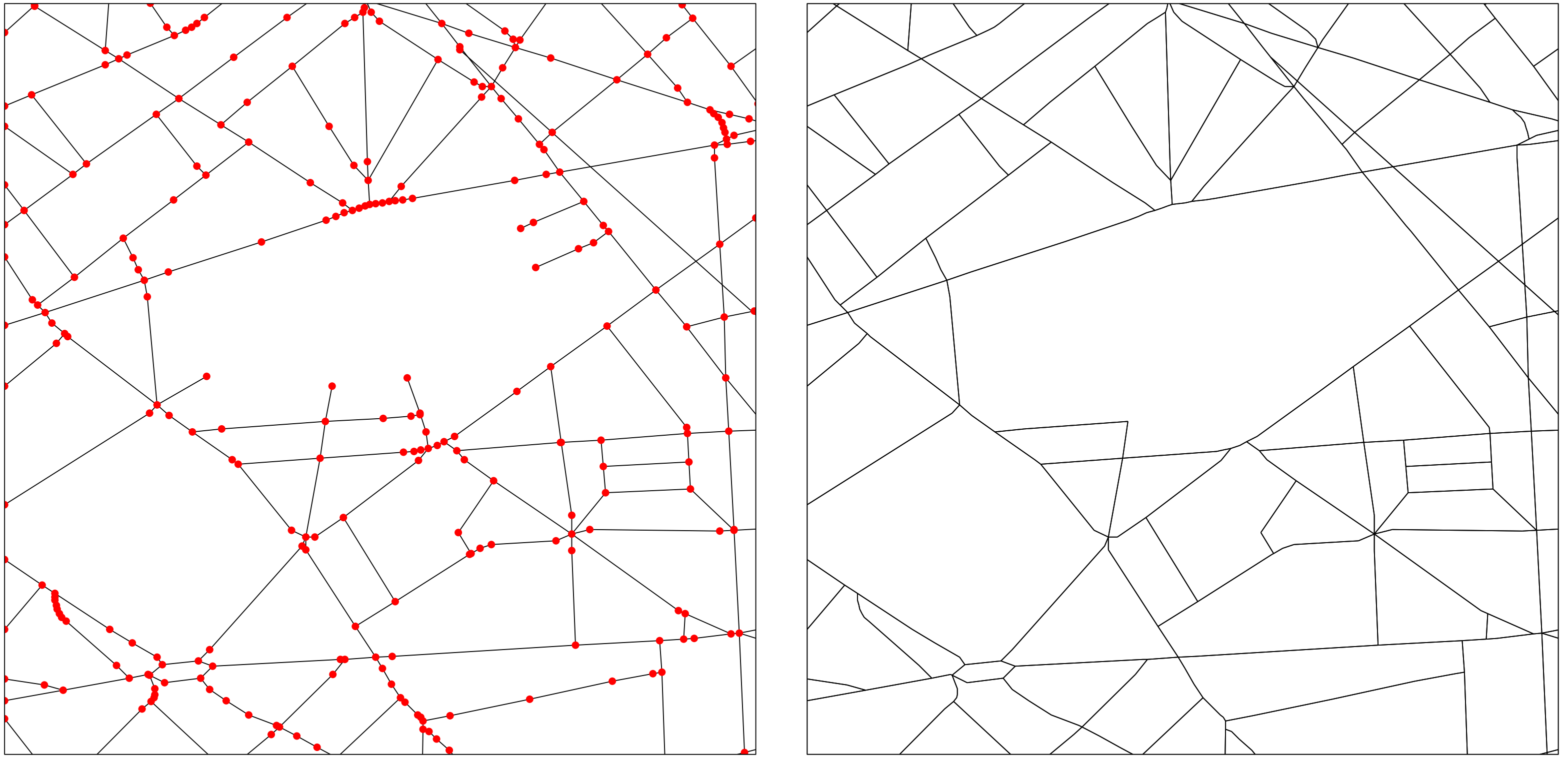
Numerical examples
Simulated input data

Solutions for problem of ambiguous decisions

- If thinning factor $p < 1$ decisions unique
  - Introduce p as additional optimization parameter
  - Increases runtime
  - Sometimes p known
- Iterative fitting procedure
  - Fitting procedure for X_0
  - Fitting of nesting with X_0 given
  - Hierarchical data structure needed

Numerical examples

Real input data



(a) *Raw data* (b) *Preprocessed data*

Data of a local region within Paris

Numerical examples

Real input data

- Decide between PLT, PVT, and PDT
- Let $\gamma \in [10^{-6}, 0.03]$, step width 10^{-8}

Tessellation	$d_{e,min}$	γ
PLT	0.29555	0.016922
PVT	0.20147	0.000065
PDT	0.77293	0.000046

Decision in favor of PVT with $\gamma = 0.000065$

Numerical examples
Real input data

- Decision for a PVT is obvious?
- Idea: Consider main roads first

Tessellation	$d_{e,min}$	γ
PLT	0.21101	0.002384
PVT	0.29749	0.000001
PDT	0.73378	0.000001

Main roads would be modeled by PLT with $\gamma = 0.002384$

Numerical examples
Real input data

Decide between PLT/PLT, PLT/PVT, and PLT/PDT

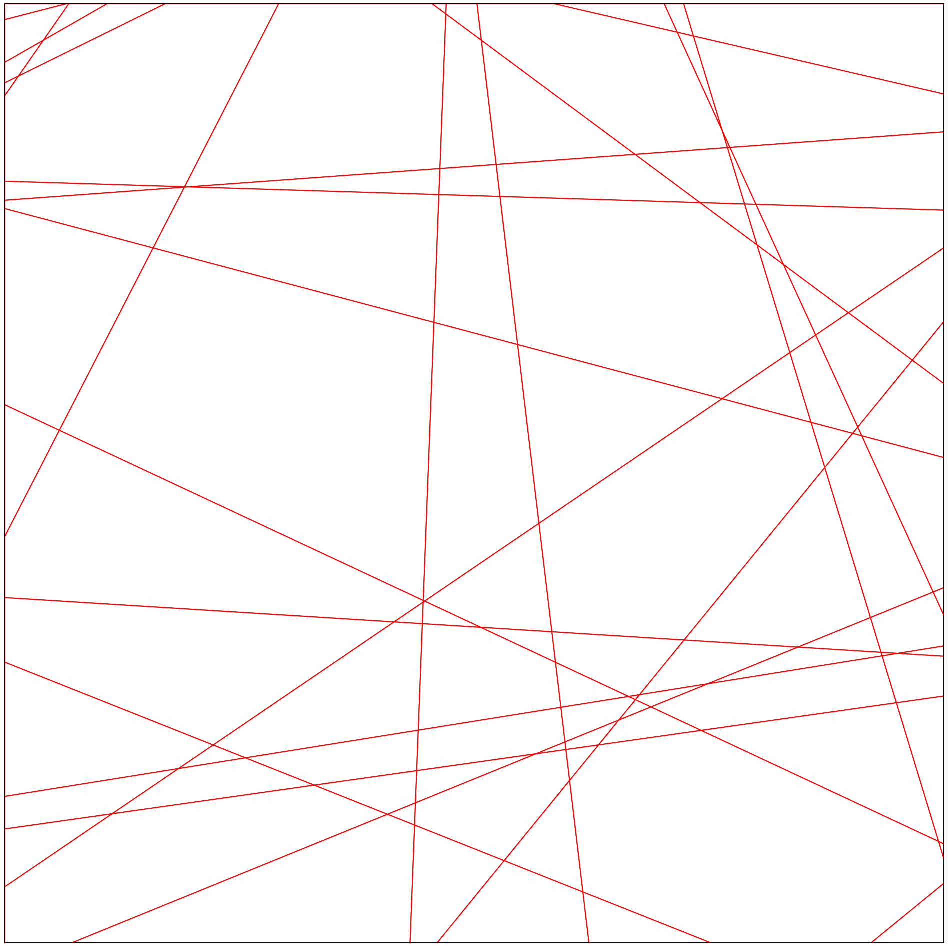
X	$d_{e,min}$	γ_0	γ_1
PLT/PLT	0.15224	0.002384	0.013906
PLT/PVT	0.20455	0.002384	0.000044
PLT/PDT	0.36649	0.002384	0.000028

Road system would be modeled by PLT/PLT–nesting with $\gamma_0 = 0.002384$ *and* $\gamma_1 = 0.013906$

Typical Cox-Voronoi cells

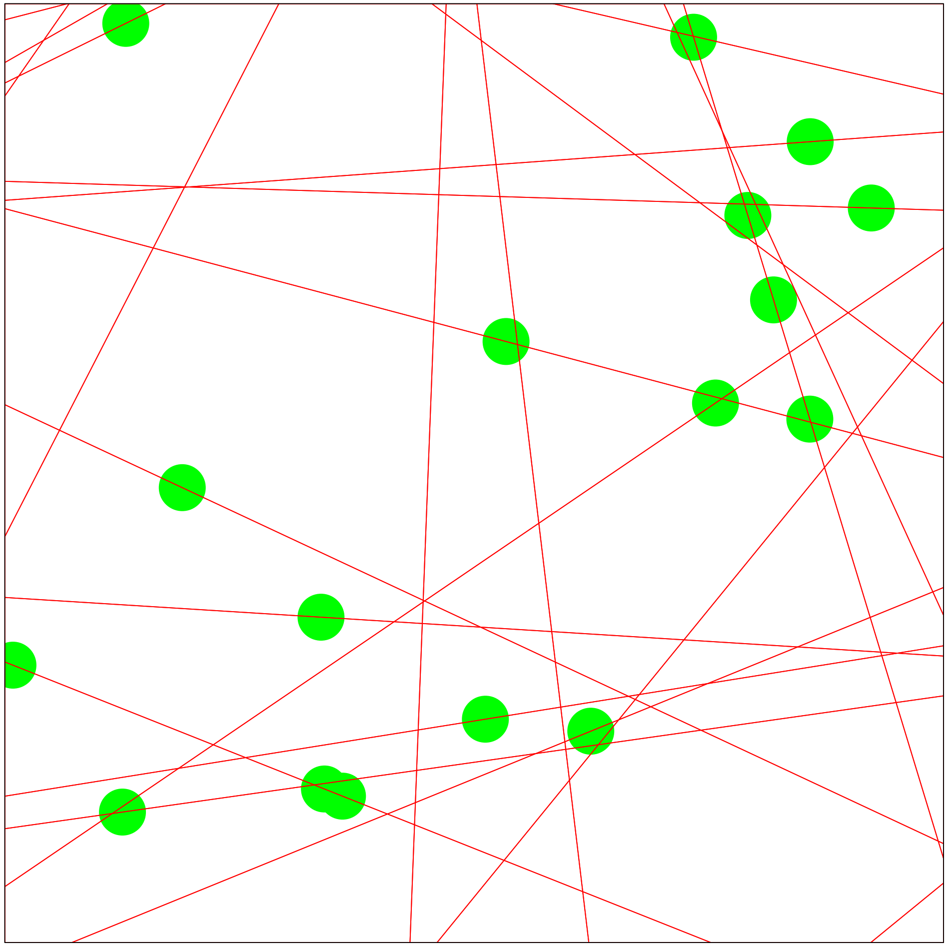
- For main roads decision mostly in favor of PLT
- Higher-level telecommuncation equipment placed on main roads only
- Placement modeled by linear Poisson processes
  - Cox processes induced by Poisson line processes
  - Typical serving zones of interest
  - Typical cells of Cox-Voronoi tessellation (CVT)
  - ’Typical’ means drawing uniformly from all cells
- Simulation algorithm based on Slyvniak’s theorem
- Usage of properties of Poisson point processes and Poisson line processes

Typical Cox-Voronoi cells



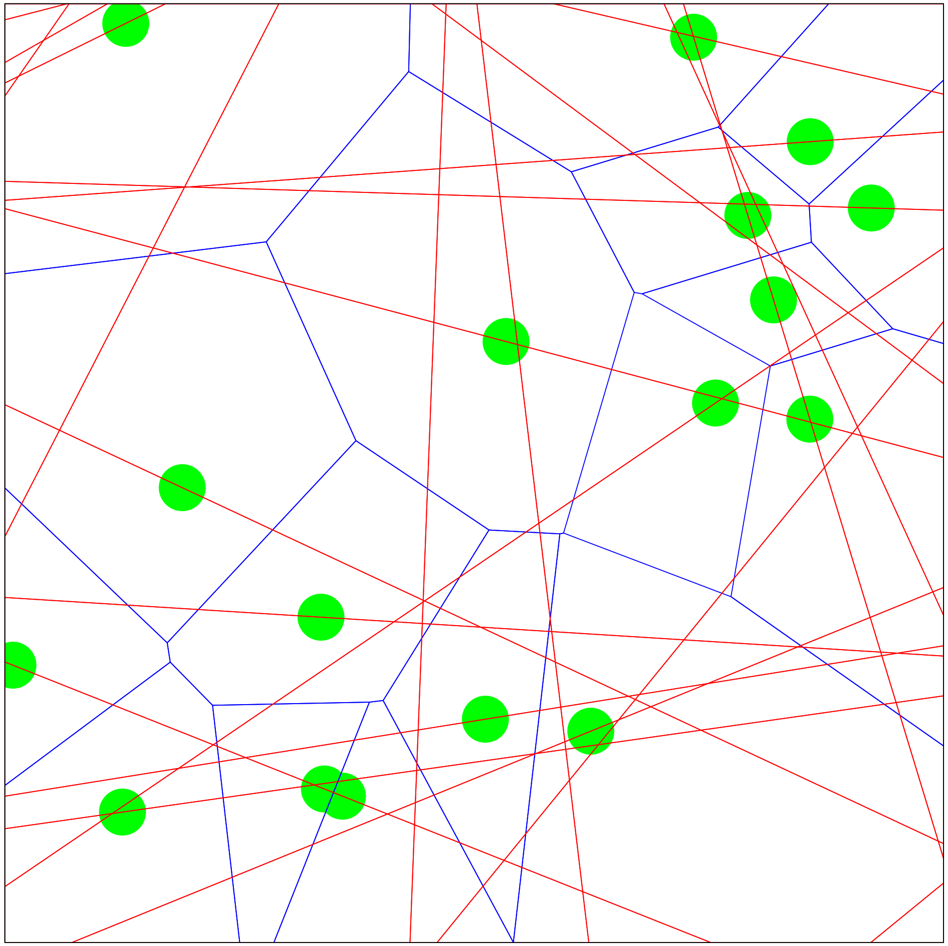
Realization of a PLT

Typical Cox-Voronoi cells



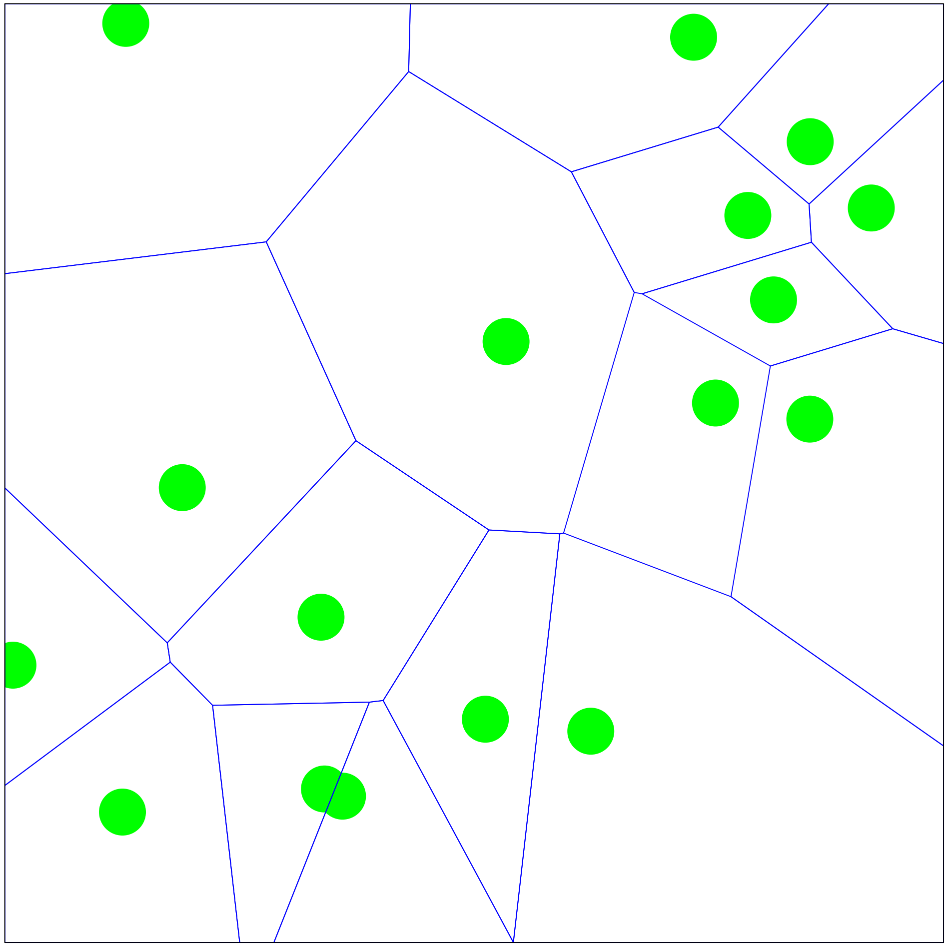
Random placement of points on PLT

Typical Cox-Voronoi cells



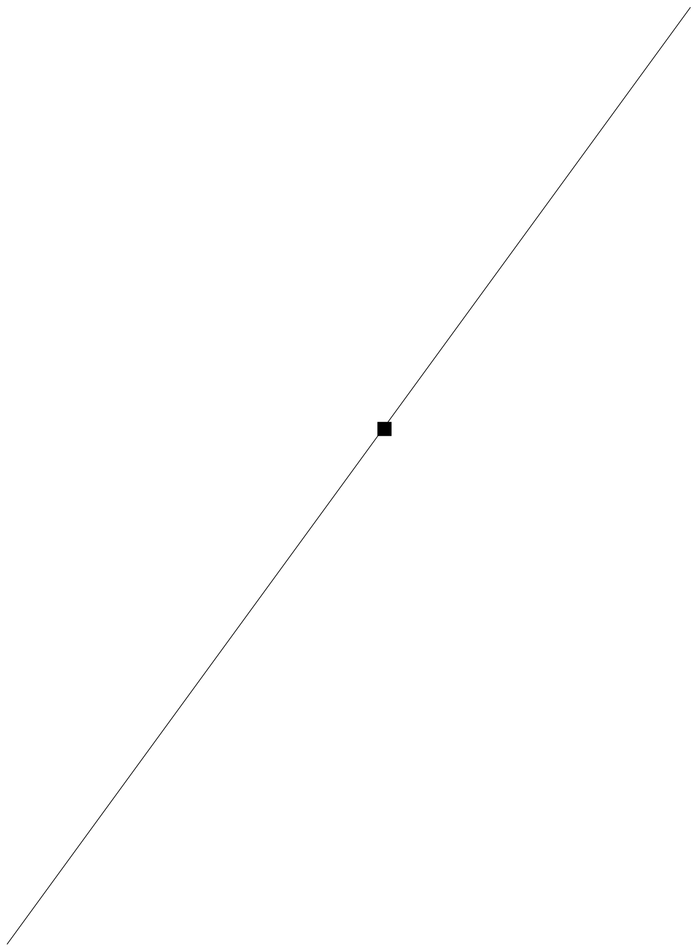
Cox-Voronoi cells

Typical Cox-Voronoi cells



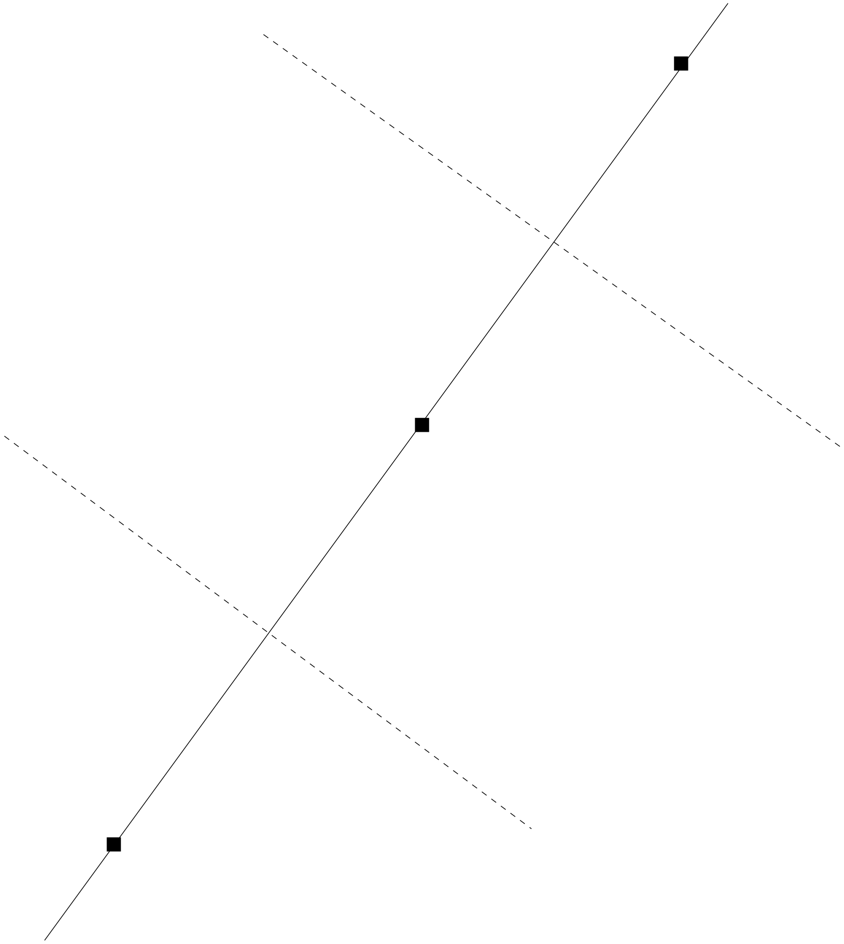
Cox-Voronoi cells

Typical Cox-Voronoi cells
Simulation algorithm



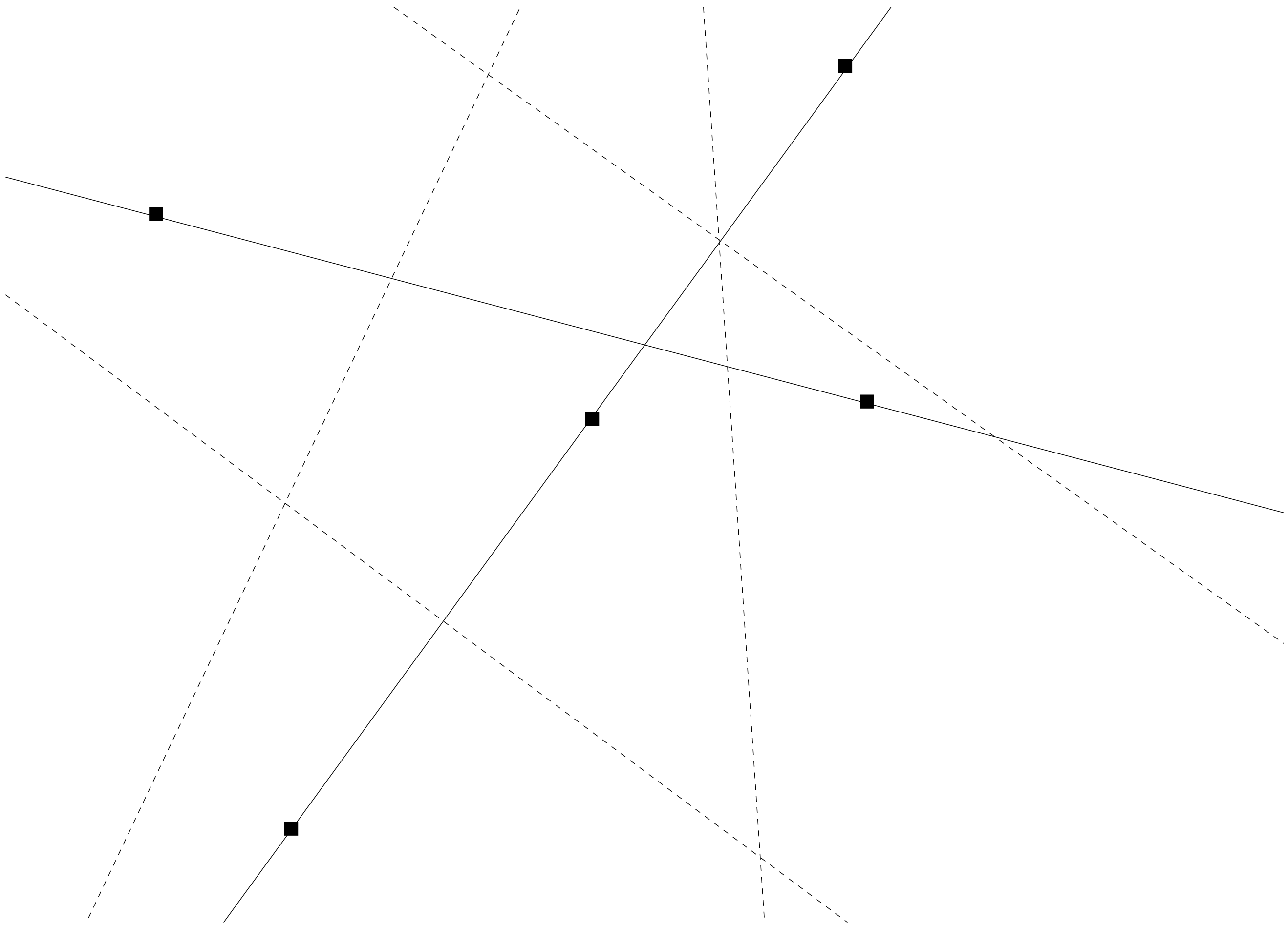
Starting line with initial nucleus

Typical Cox-Voronoi cells
Simulation algorithm



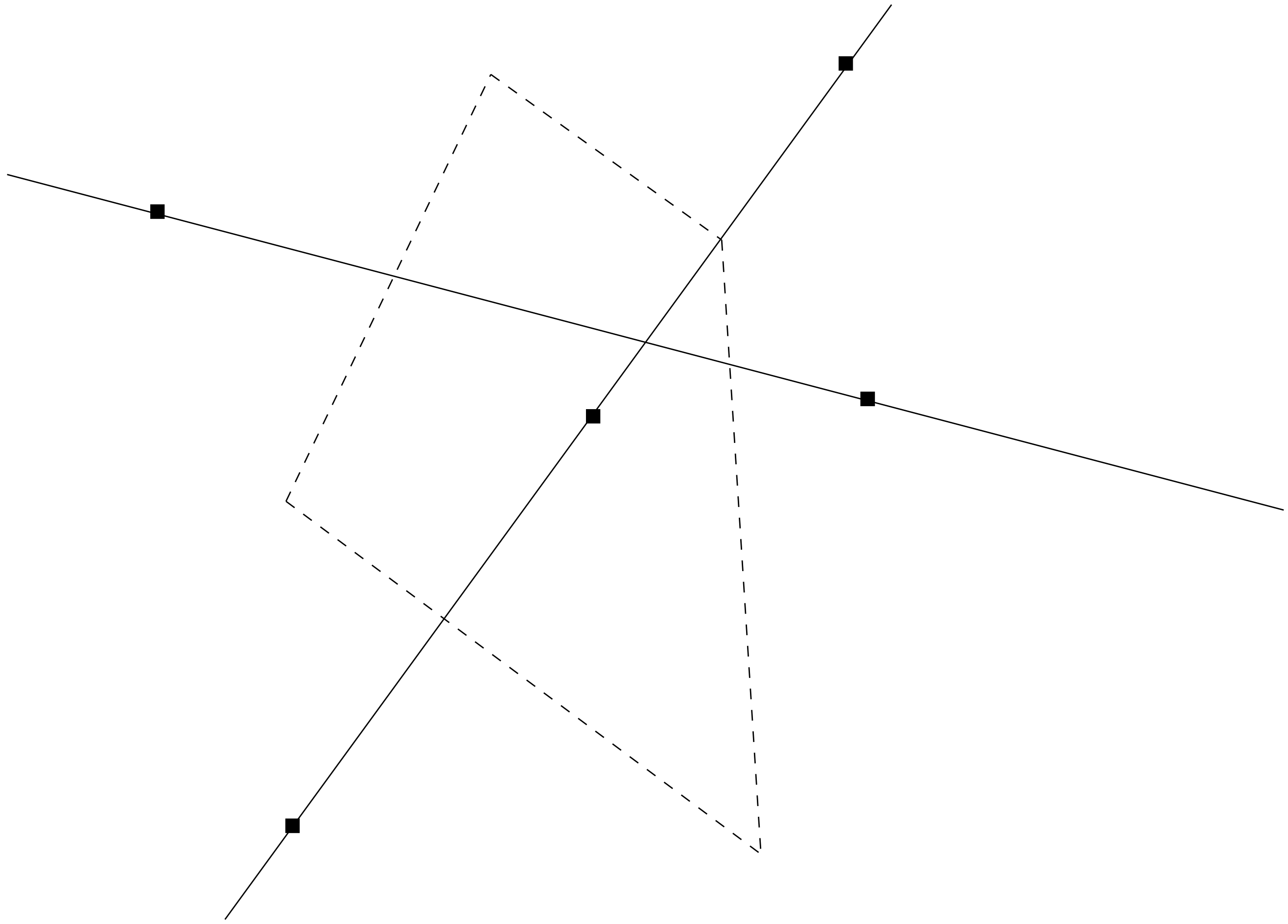
Placement of neighboring nuclei

Typical Cox-Voronoi cells
Simulation algorithm



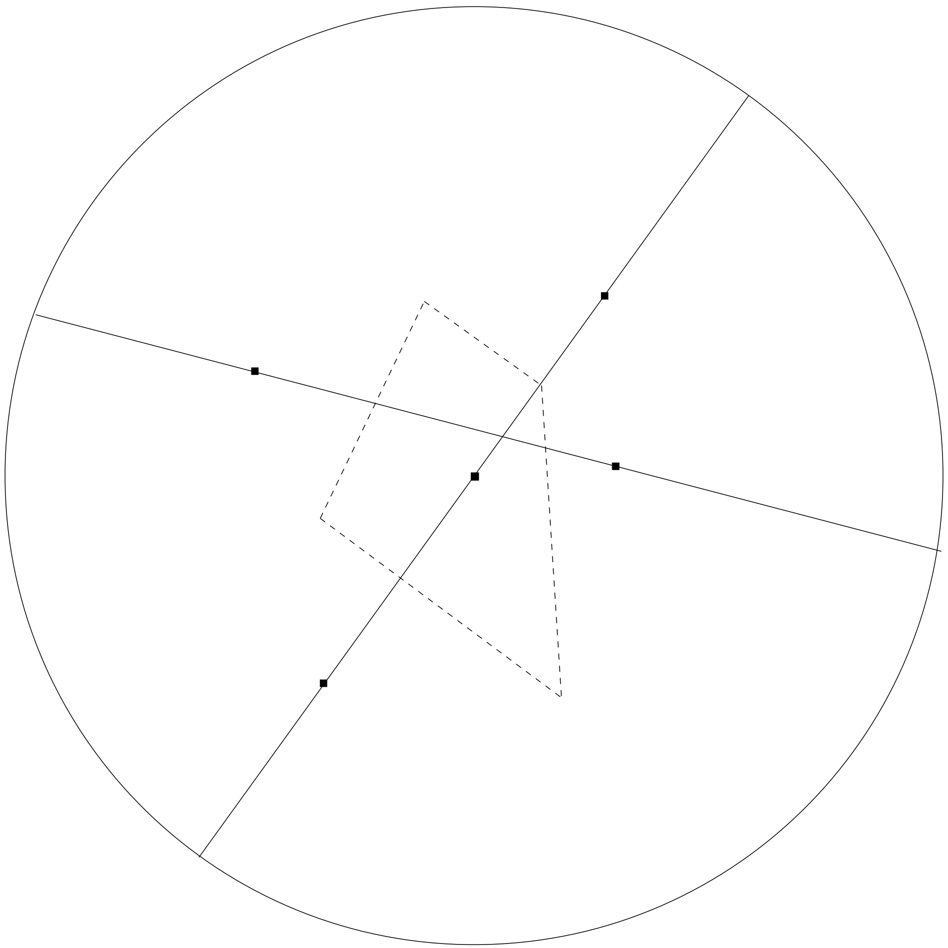
Second line with points on it

Typical Cox-Voronoi cells
Simulation algorithm



Initial cell

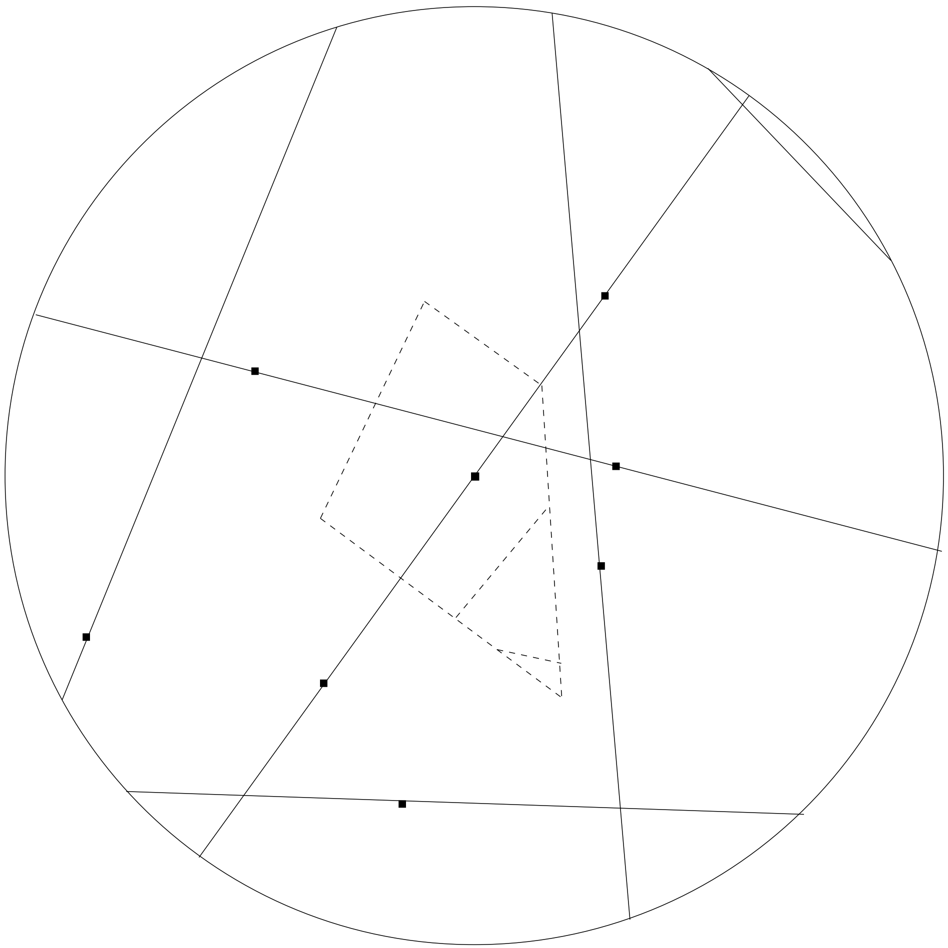
Typical Cox-Voronoi cells
Simulation algorithm



Stopping criterion

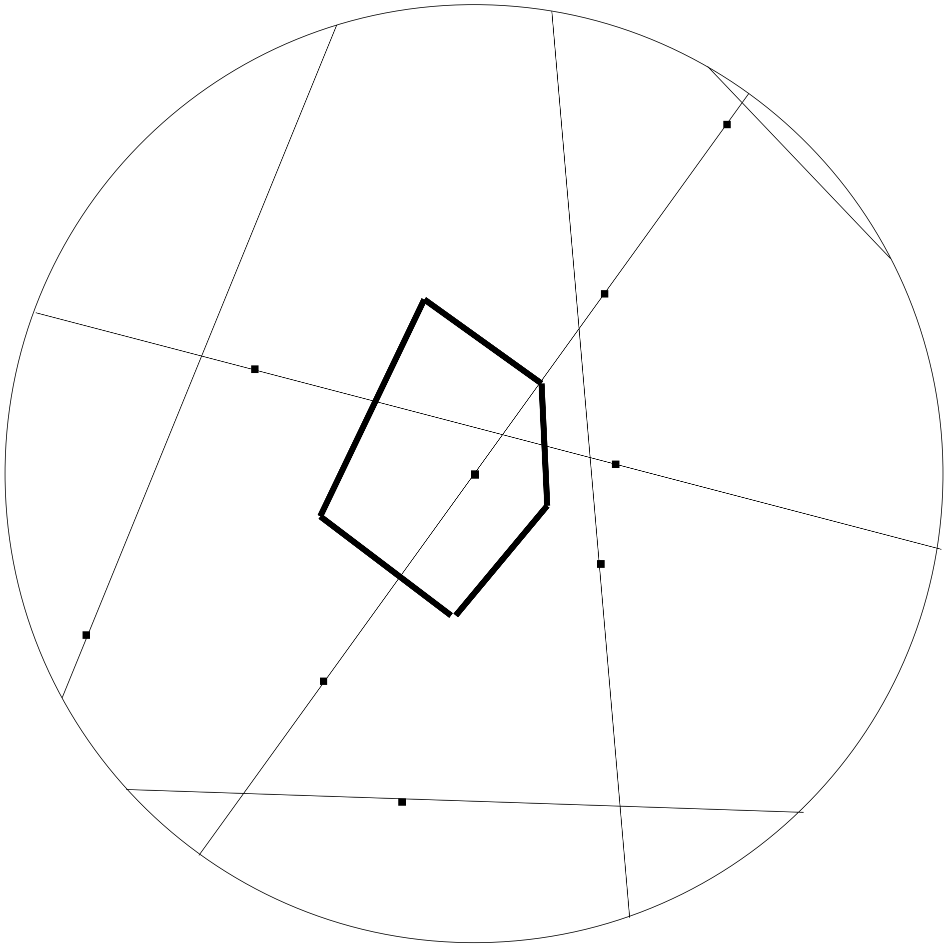
Typical Cox-Voronoi cells

Simulation algorithm



Cutting the initial cell

Typical Cox-Voronoi cells
Simulation algorithm

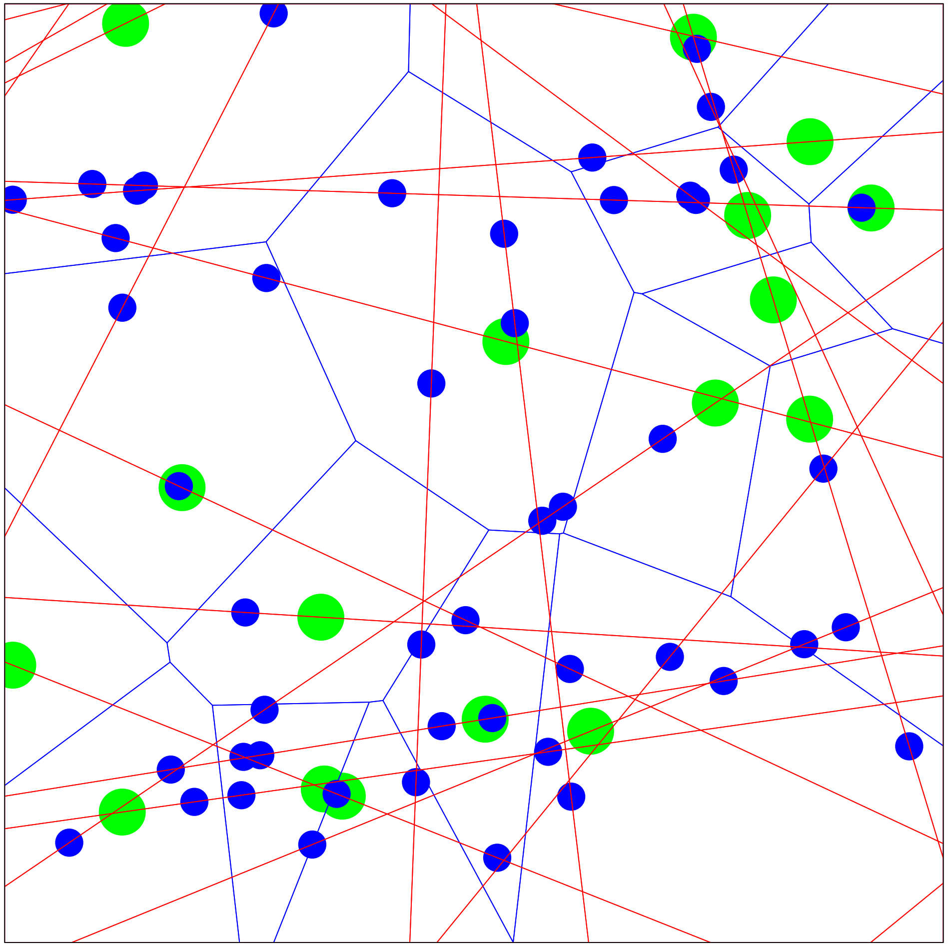


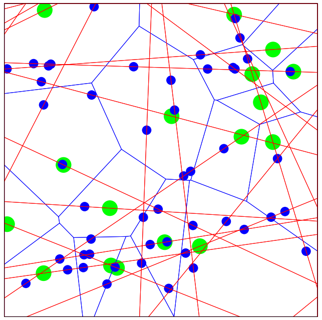
Typical Cox-Voronoi cell

Typical Cox-Voronoi cells
Scaling property

- Typical cell can be analyzed regarding geometric characteristics (e.g. area, perimeter, number of vertices, shape,..)
- Model parameters
  - γ (intensity of line tessellation)
  - λ_C (intensity of components on the lines)
  - $\lambda = \gamma\lambda_C$ (intensity of the point process with respect to unit area)
- Scaling property
  - Same structure but on a different scale
  - $\kappa = \gamma/\lambda_C$ important parameter
  - $1/\gamma$ good measure for scale

Typical Cox-Voronoi cells
Scaling property





Scaling property, different intensities but same κ

Typical Cox-Voronoi cells

Results

- Estimations for first order and second order moments of geometric characteristics for any given pair (γ^*, λ_C^*)
- Information about distribution of geometric characteristics (=> risk analysis)
- Similarities to typical cell of Poisson-Voronoi tessellation, especially for large κ
- Useful for simulation of network characteristics
  - Mean shortest path lengths
  - Mean subscriber line lengths

Literature

- C. Gloaguen, F. Fleischer, H. Schmidt and V. Schmidt (2004)
  Fitting of stochastic telecommunication network models via distance measures and Monte-Carlo tests.
  Preprint, submitted to *Telecommunication Systems*
- C. Gloaguen, F. Fleischer, H. Schmidt and V. Schmidt (2005)
  Simulation of typical Cox-Voronoi cells, with a special regard to implementation tests.
  Preprint, submitted to *Mathematical Methods of Operations Research*
- M. Beil, H. Braxmeier, F. Fleischer, V. Schmidt and P. Walther (2005)
  Quantitative analysis of keratin fi lament networks in scanning electron microscopy images of cancer cells.
  Preprint, submitted to *Journal of Microscopy*

For further information please visit
http://www.geostoch.de

Thank you for your attention!