M. Hadac, S. Hew

$$\partial_x(\partial_t u + u_{xxx} + u u_x) + u_{yy} = 0$$

Kadomtsev - Petviashvili '70

Symmetries

① Translation invariant

② Scaling $\lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$

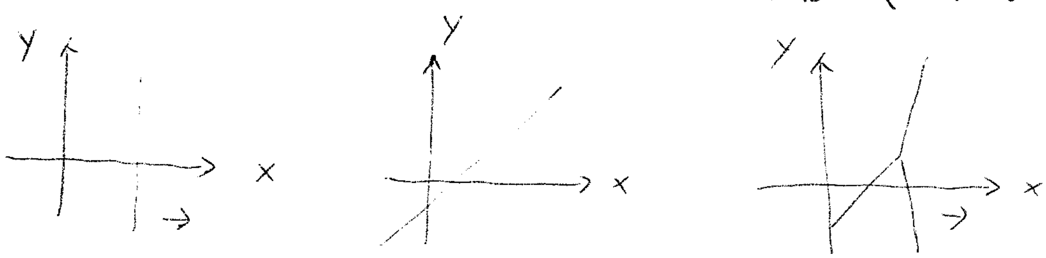
③ Galilean inv $u(t, x - cy - c^2 t, y + 2ct)$

Lax pair

Integrals $I_0 = \frac{1}{2} \int u^2 \, dx\, dy$

$$I_1 = \frac{1}{2} \int (\partial_x u)^2 - \frac{1}{3} u^3 - (\partial_x^{-1} \partial_y u)^2 \, dx\, dy$$

Formula for solutions : Like solitons (sol to kdV)



Natural space for initial data:

$H^{s,0}_{x,y}(\mathbb{R}^2) = \langle D_x \rangle^{-s} L^2$

critical space $\dot{H}^{-1/2,0} = |D_x|^{1/2} L^2$

Wellposedness : Bourgain '93 L^2

local
- Takaoka '00 $\dot{H}^{s,0}$ $s > -\frac{1}{2}$
- Tak & Tzvetkov $H^{s,0}$ $s > -\frac{1}{3}$
- Hadac '07 $s > -\frac{1}{2}$

Thm: Local wellposedness in $H^{-\frac{1}{2},0}$, $\dot{H}^{-\frac{1}{2},0}$

- global w.p in $H^{-\frac{1}{2},0}$ small data
- scattering for small data in $\dot{H}^{-\frac{1}{2},0}$

Linear estimates

$$(u_t + u_{xxx})_x + u_{yy} = 0$$

define a group $S(t)$

$$\| S(\cdot) v \|_{L^4(\mathbb{R}\times\mathbb{R}^2)} \lesssim \| v \|_{L^2(\mathbb{R}^2)}$$

$$\| (P_{N_1} S v_1)(P_{N_2} S v_2) \|_{L^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} \| v_1 \|_{L^2(\mathbb{R}^2)} \| v_2 \|_{L^2(\mathbb{R}^2)}$$

$$\| \partial_x S(\cdot) v \|_{L^\infty_x L^2_{ty}} \lesssim \| v \|_{L^2}$$

Bourgain spaces

$$\| u \|_{\dot{X}^{s,b}} = \| D_x^s D_t^b S(-t) u(t) \|_{L^2(\mathbb{R}^3)}$$

critical space $\dot{X}^{-\frac{1}{2},\frac{1}{2}}$, $\dot{X}^{-\frac{1}{2},-\frac{1}{2}}$

$$\| u \|_{L^4(\mathbb{R}^3)} \lesssim \| u \|_{X^{0,\frac{1}{2}}}$$

$$\| P_{N_1} u_1 P_{N_2} u_2 \|_{L^2} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{4}} \| u_1 \|_{\dot{X}^{0,\frac{1}{2}}} \| u_2 \|_{\dot{X}^{0,\frac{1}{2}}}$$

$$\| \partial_x u \|_{L^\infty_x L^2_{ty}} \lesssim \| u \|_{X^{0,\frac{1}{2}}}$$ almost true

$$\| f \|_{\dot{X}^{0,-\frac{1}{2}}} \lesssim \| f \|_{L^{4/3}}$$

$$\| u \|_{\dot{X}^{0,\frac{1}{2}}} \lesssim \| u(0) \|_{L^2} + \| (\partial_t + \partial^3_{xxx} + \partial_x^{-1} \partial_y) u \|_{\dot{X}^{0,-\frac{1}{2}}}$$

proof: $X = \dot{X}^{-\frac{1}{2},\frac{1}{2}}$, $Y = \dot{X}^{-\frac{1}{2},-\frac{1}{2}}$

2. Lemma $\| \partial_x(u v) \|_Y \leq \| u \|_X \| v \|_X$

3. Lemma $$\left| \int_0^T \int_{\mathbb{R}^2} u_{N_1} v_{N_2} w_{N_3} \, dx\, dy\, dt \right| \lesssim \frac{C}{(N_1 N_2 N_3)^{\frac{1}{2}}} \| u_{N_1} \|_{\dot{X}^{0,\frac{1}{2}}} \| v_{N_2} \|_{\dot{X}^{0,\frac{1}{2}}} \| w_{N_3} \|_{\dot{X}^{0,\frac{1}{2}}}$$

$$\lambda_i = \tau_i - \xi_i^3 + \frac{y_i^2}{\xi_i} \qquad i = 1, 2, 3$$

$$\xi_1 + \xi_2 + \xi_3 = 0 \quad , \quad \tau_1 + \tau_2 + \tau_3 = 0$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 3\,\xi_1 \xi_2 \xi_3 + \frac{(\xi_2 y_1 - y_2 \xi_1)^2}{\xi_1 \xi_2 \xi_3} \qquad \text{resonance}$$

$$\Longrightarrow \max \{ |\lambda_i| \} \geq |\xi_1 \xi_2 \xi_3| \qquad \text{suppose} \qquad |\lambda_i| = \max\{|\lambda|, 1\}$$

$$\Big| \int_{|\lambda_1| = \max\{|\lambda_i|\}} u_{N_1} u_{N_2} u_{N_3} \, dx\,dy\,dt \Big| \leq \underbrace{\| u_{N_1} \|_{L^2}}_{(N_1 N_2 N_3)^{-1/2}} \| u_{N_2} u_{N_3} \|_{L^2}$$

$$\Downarrow$$

$$\| u_{N_2} \|_{\dot X^{0,1/2}} \| u_{N_3} \|_{\dot X^{0,1/2}}$$

proof lemma 2

$$\| \partial_x (uv) \|_Y^2 \sim \sum_N \| P_N \partial_x uv \|_Y^2 \lesssim \sum_N N^{-1} \| P_N \partial_x (uv) \|_{\dot X^{0,1/2}}$$

$$\lesssim \sum_{N_1, N_2} \| P_{N_1} u \|^2_{\dot X^{0,1/2}} \, N_2' \, \| P_{N_2} u \|^2_{\dot X^{0,1/2}} + \text{similar terms}$$

$$\lesssim \| u \|_{\dot X^{-1/2,1/2}} \, \| v \|_{\dot X^{-1/2,1/2}}$$

Wiener · Bounded p-variation, Tataru, K. T.

$$\| u \|_{V^p} = \sup_{t_0 < t_1 < \cdots < t_n} \sum_{j=1}^{n} \| u(t_j) - u(t_{j-1}) \|^p$$

Atoms $a = \sum_j a_j \, \chi_{[t_{j-1}, t_j)}$, $\sum \|a_j\|^p = 1$

Atomic spaces U^p

$V^p \subset V^q$, $U^p \subset U^q$, $U^p \subset V^p$, $p < q$

$(U^p)^* = V^q$

$$V^p_{rc} \subset V^p$$

$$\|u\|^2_{X^s} = \sum_N N^{2s} \| S(-t)\, u(t) \|^2_{U^2}$$

$$\| f\|_{X^s} = \sum_N N^{2s} \| \partial_t S(-t) f(t) \|^2_{U^2}$$

Strichartz estimates

$$\| u\|_{L^4} \lesssim c \| S(-t)\, u\|_{L^4} \lesssim \| S(-t)\, u \|_{V^2_{rc}}$$

bilinear estimates

$$\| u_{N_1} u_{N_2} \|_{L^2} \leq c \left(\frac{N_1}{N_2} \right)^{1/4} \| S(t)\, u_{N_1} \|_{U^2} \| S(t)\, u_{N_2} \|_{U^2}$$

$$\| u_{N_1} u_{N_2} \|_{L^2} \lesssim \left(\frac{N_1}{N_2} \right)^{1/4} \left(1 + \ln \left(1 + \frac{N_2}{N_1} \right) \right)^2$$

$$\| S(-t)\, u_{N_1} \|_{V^2} \| S(-t)\, u_{N_2} \|_{V^2}$$

$$\| u \|_{X^{0,1/2,\infty}} \lesssim \| S(-t)\, u \|_{V^2} \lesssim \| S(-t)\, v \|_{U^2} \lesssim \| u \|_{X^{0,1/2,1}}$$