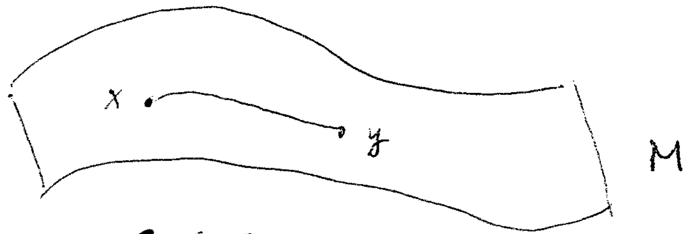
Joint work with Topping    www.math.toronto.edu/mccann

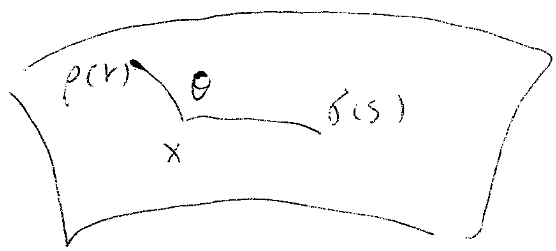
$(M^n, g_{ij}(t))$




$\sigma : s \in [0,1] \to M$

$$d^2(x,y,t) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} \int_0^1 g_{ij}(\sigma(s),t) \frac{d\sigma^i}{ds} \frac{d\sigma^j}{ds}\, ds$$

$$d(\sigma(s), \sigma(s'), t) = |s-s'|\, d(x,y,t) \quad \forall\, s, s' \in [0,1]$$




$$d^2(\sigma(s), \rho(r), 0)$$

$$= s^2 + r^2 - 2 \left\langle \frac{d\sigma}{ds}, \frac{d\rho}{dr} \right\rangle sr$$

$$- \frac{k}{6} s^2 r^2 + O\left( (s^2+r^2)^{5/2} \right) \text{ as } s, r \to 0 \qquad \left|\frac{d\sigma}{ds}\right| = 1 = \left|\frac{d\rho}{dr}\right|$$

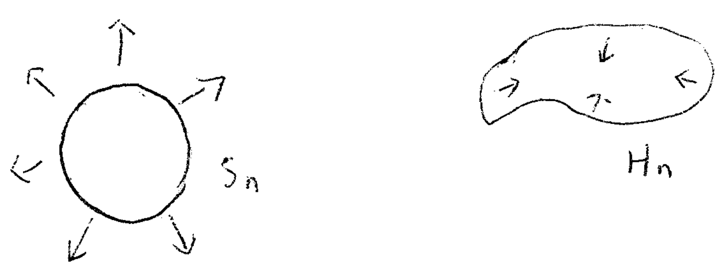
when $\quad k = \sec_x(\dot\sigma(0), \dot\rho(0), \dot\sigma(0), \dot\rho(0))$

$$Ric(\dot\sigma, \dot\sigma) = (n-1) \langle \sec \rangle_{\dot\rho \perp \dot\sigma}$$

$$= Ric_{ij}\, \dot\sigma^i(0)\, \sigma^j(0)$$

$$\frac{\partial g_{ij}}{\partial t} = +2 Ric_{ij} \qquad \text{backward Ricci flow}$$

e.g.



~~[illegible]~~ longtime behavior? singularity? self-similiar ~~[illegible]~~ ?

Recall: heat equation in a fixed metric or even on Euclidean space. ~~[illegible]~~ $0 \le w$ ~~[illegible]~~ $\in L^1(\mathbb{R}^4)$

$L^1(M, g_{ij}(0))$

$$\frac{\partial w}{\partial t} = \Delta w \quad , \quad I = \int_M dw$$

$(M, d(x,y,t))$ Given two probability measures $w, \tilde{w} \ge 0$ on $M$, ~~[illegible]~~ consider joint $0 \le \gamma$ on $M \times M$ $\ni$ $w(U) = \gamma(U \times M)$, $\gamma(M \times U) = \tilde{u}(U)$

$$W^2(w, \tilde{w}, t) = \inf_{\substack{0 \le \gamma \\ w \quad \tilde{w}}} \int_{M,M} d^2(x,y,t)\, d\gamma(x,y)$$

$W(\tilde{w}, w, t)$ is a distance function on probabilities $w, \tilde{w} \in \mathcal{P}(M)$

$\mathcal{P}(M)$ is a geodesic space if $M$ is; $\mathcal{P}(M)$ is a formal $\infty$ dim Riemannian manifold, if $M$ is a finite dim one. Otto

Thm: any two solution $w(t)$, $\hat{w}(t)$ with unit mass satisfies

$W(w(t), \hat{w}(t), 0) \le W(w(0), \hat{w}(0), 0)$ iff $Ric_M \ge 0$

(weak definition of $Ric \ge 0$)

$(M^n, g_{ij}(t))$ ; Consider $n$-forms $\omega(t), \hat{\omega}(t) \geq 0$

evolving under $\frac{\partial \omega}{\partial t} = \Delta_{g(t)} \omega$
connection Laplacian

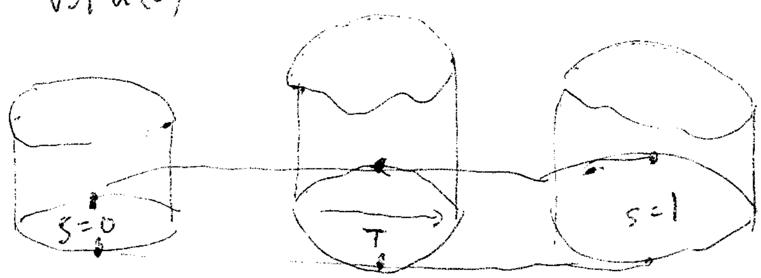
Ideas of proof:

Otto: heat eq represents steppest descent of

$$E(\omega) = \int_M \frac{d\omega}{d\mathrm{vol}} \log \frac{d\omega}{d\mathrm{vol}} \, d\mathrm{vol} \quad \text{w.r.t distance } W(u, \hat{u}, 0)$$

CMCS: $Ric \geq 0 \Rightarrow \frac{d^2}{ds^2} E(\omega'(s)) \geq 0$ if $W(\omega(s), \omega(s'), 0)$
$= |s-s'| \, W(\omega(0), \omega(1), 0)$

$$\omega(0) = \frac{\chi_{u(0)}}{\mathrm{vol}\, u(0)} \qquad \omega(1) = \frac{\chi_{u(1)}}{\mathrm{vol}\, u(1)}$$




Thm: If $(M^n, g_{ij}(t))$ is smooth then $\frac{\partial g_{ij}}{\partial t} \leq 2 Ric_{ij}$

holds if and only if $W(\omega(t), \tilde{\omega}(t), t) \leq W(\omega(0), \tilde{\omega}(0), 0)$

for all $n$ forms evolved by $\frac{\partial \omega}{\partial t} = \Delta \omega$

Alternating, given $(M, d(x,y,t), d\mathrm{vol}(t))$

define $E(\omega, t) = \int_M \frac{d\omega(t)}{d\mathrm{vol}(t)} \log \frac{d\omega(t)}{d\mathrm{vol}(t)} \, d\mathrm{vol}(t)$

Thm: If $(M^n, g^{ij}(t))$ is smooth and $dvol(t) = \sqrt{\det g_{ij}(t)}\ d^n x$

then $\frac{\partial g_{ij}}{\partial t} \leq 2\, Ric_{ij}$ holds iff $\forall$ $w(0), w(1) \in \mathcal{P}(M)$

$$\frac{d}{dt} \frac{W^2}{2}(w(0), \hat{w}(1); t) \leq \int_0^1 ds\ \frac{d^2 E}{ds^2}(w(s), t))\ ds$$

$\hat{\alpha}(1)$, $d\widehat{vol}(1)$

defines a subsolution to the reverse Ricci flow

a solution $\alpha(t)$, $dvol(t)$ would be a maximal subsolution

in the sense that

$$\frac{d}{dt} \alpha(x, y, t_0) \geq \frac{d}{dt} \hat{\alpha}(x, y, t_0)$$

$\forall$ subsolution $\hat{\alpha}$,

$\hat{\alpha} = \alpha$ at $t_0$

$\widehat{vol} = vol$ at $t_0$