w/ J Marzuola and D Tataru

$$(i\partial_t + \Delta) u = 0 \qquad (t, x) \in \mathbb{R}\times\mathbb{R}^n$$
$$u(0) = u_0$$

Smoothing $\quad \| \langle x\rangle^{-\frac{1}{2}-} D^{\frac{1}{2}} u \|_{L^2_{t,x}} \lesssim \|u_0\|_2$

Strichartz $\quad \| u\|_{L^p_t L^q_x} \lesssim \|u_0\|_2 \quad ; \quad 2 \le p, q \le \infty$

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$$
$$(n, p, q) \neq (2, 2, \infty)$$

Variable Coeff . $\quad D = \frac{\partial}{i}$

$$[D_t + \underbrace{(D_i a^{ij}(t,x) D_j + D_i b^i(t,x) + b^i D_i(t,x) + c(t,x))}_{A(t,x,D)}] u = f \qquad u(0) = u_0$$

Tataru : long range perturbations

$$\sum_j \sup_{|x| \approx 2^j} \langle x\rangle^2 \left(|\nabla_x^2 a| + |\partial_t a|\right) + \langle x\rangle \left( |\nabla_x a| + |a - I|\right) \le K$$

- Smoothing $\Rightarrow$ Strichartz
- $K = \varepsilon \Rightarrow$ Smoothing

lower oder term :

$$\sum_j \sup_{|x| \approx 2^j} \langle x\rangle |b| \le K \qquad \text{Take } n \geq 3$$

$$\sup_{|x| \approx 2^j} \langle x\rangle^2 \left( |c| + |\operatorname{div} b| \right) \le K$$

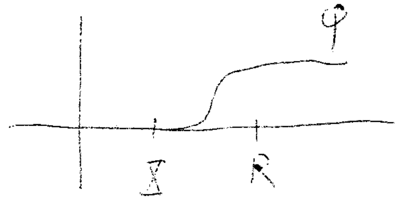
$$\limsup_{|x| \to \infty} \langle x\rangle^2 ( |c| + |\operatorname{div} b|) \le \varepsilon \ll 1$$

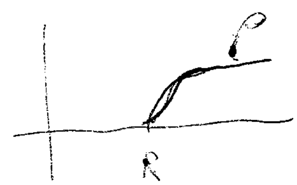
Local smoothing spaces:

$$\|u\|^2_{\tilde{X}} = \|\langle x\rangle^{-1} u\|^2_{L^2_{t,x}} + \sum_{k=-\infty}^{\infty} 2^k \|S_k u\|^2_{X}$$

$$\|u\|_{X_k} = \sup_{j \geq \max\{0,-k\}} \|\langle x\rangle^{-1/2} u\|_{L^2_{t,x}(|x| \approx 2^j)} + \min(1, 2^{-k/2}) \|u\|_{L^2_{t,x}(|x| < \max(1, 2^{-k}))}$$

Thm: $\exists\, X, R$




$$\|\varphi u\|_{\tilde{X}} \lesssim \|u_0\|_2 + \|f\|_{\tilde{X}'} + \|u\|_{L^2_{t,x}(|x|<R)}$$

$$\|\rho u\|_{L^2_t L^q_x} \lesssim \|u_0\|_2 + \|f\|_{\tilde{L}^{p'} L^{q'}} + \|u\|_{L^2_{t,x}(|x|<R)}$$

Low/med — $Q \approx Q(r) \underbrace{f(r)}_{\text{bdd}} \partial_r$

High fre — $Q \approx Q(r) f(r) \dfrac{x_i a^{ij} \partial_j}{r}$

Nontrapping

Thm: $\|u\|_{\tilde{X} \cap L^p_t L^q_x} \lesssim \|u_0\|_2 + \|f\|_{\tilde{X}' + \tilde{L}^{p'} L^{q'}} + \|u\|_{L^2_{t,x}(|x|<2R)}$

Prop. nontrapping $\Rightarrow \exists$ real symbol s.t. $H_a\, q \geq |\xi|$ in $|x| < 2R$

Time independent, nontrapping, Prove estimates $P_c u$

$\sigma_c = [0, \infty)$

Thm time ind, nontrapping $0$ is not an e-value or resonance of $A$

$$\Rightarrow \|P_c u\|_{\tilde{\hat{X}} \cap L_t^p L_x^q} \lesssim \|u\|_2 + \|f\|_{\tilde{\hat{X}}' + L_t^{p'} L_x^{q'}}$$

Proof: WLOG $u_0 = 0$, Look at $(D_t + A - i\varepsilon) u_\varepsilon = f$

$\tilde{\hat{X}} = L_t^2 \tilde{\hat{X}}^0$, $v(\tau) = \hat{u}(\tau)$

Know: $\|v\|_{\tilde{\hat{X}}^0} \lesssim \|(A - \tau - i\varepsilon) v\|_{\tilde{\hat{X}}^{0\prime}} + \|v\|_{L^2(|x| < 2R)}$

Goal $\|P_c v\|_{\tilde{\hat{X}}^0} \lesssim \|(A - \tau - i\varepsilon) v\|_{\tilde{\hat{X}}^{0\prime}}$

- $|\tau|$ large: Elliptic regularity $\Rightarrow \tau^{1/4} \|v\|_{L^2(|x| \le 2R)} \lesssim \|v\|_{\hat{X}^0} + \|(A - \tau - i\varepsilon) v\|_{\hat{X}^{0\prime}}$

- $|\tau| < C$, Suppose not.

There exists $\varepsilon_n \to 0$, $\tau_n \to \tau$, $v_n \in \tilde{X}^0$, $v_n = P_c v_n$

$\|(A - \tau_n - i\varepsilon_n) v_n\|_{\tilde{X}^{0\prime}} \to 0$, $\|v_n\|_{L^2(|x| < 2R)} = 1$

Let $v_n \to v$, $v \in \tilde{X}^0$, $P_c v = v$, $\underbrace{(A - \tau) v = 0}_{\Rightarrow v \text{ is e fun or resonance}}$, $\|v\|_{L^2(|x| < R)} = 1$

Show $v \in L^2$ then done

• $\tau < 0$

$$\|v\|_{H^1}^2 \lesssim \langle (D_i a^{ij} D_j - \tau) v, v \rangle$$

$$\lesssim \|v\|^2_{\tilde{\underline{X}}^0} < \infty$$

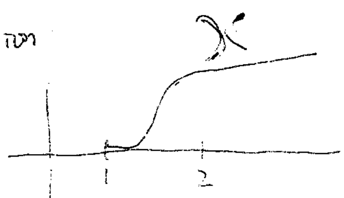
$\Rightarrow \quad v \in L^2$

• $\tau > 0$ Take $A = -\Delta$

" Outgoing radiation condition "

$$\lim_{j \to \infty} \| r^{-1/2} (\partial_r - i\tau^{1/2}) v \|_{L^2(|x| \approx 2^j)} = 0$$

Application



$$\chi_j = \chi\left(\frac{\cdot}{2^j}\right)$$

① $$0 = \langle [A, \chi_j] v, v \rangle = \mathrm{Im} \langle 2^{-j} \chi'(2^{-j} x)(\partial_r - i\tau^{1/2}) v, v \rangle$$
$$+ 2^j \tau^{1/2} \langle \chi' v, v \rangle$$

$$\| r^{-1/2} v \|_{L^2(|x| < 2^j)} \to 0$$

$$\| r^{-1/2} \nabla v \|_{L^2(|x| \gtrsim 2^j)} \to 0$$

② $$Q = b(r)(\partial_r - i\tau^{1/2}) \qquad b \approx \begin{cases} 1 & r > 2^j \\ (2^j r)^\delta & r < 2^j \end{cases}$$

$$0 = 2\,\mathrm{Re} \langle Qu, (A - \tau \cancel{\ldots}) u \rangle = \langle [A, Q] u, u ] + \cancel{\ldots}$$

$$\cancel{= -\partial_\theta^2 b'(r)(\partial_r - i\tau^{1/2})^2 = a'(A-\tau) + \text{errs}}$$

$$= -r^{-3} \chi \partial_\theta^2 - \chi' \partial_r^2 + O(\chi r^{-3})$$

$$\int \tau^{-3} \chi (\partial_\theta v)^2 + \chi' (\partial_r v)^2 \lesssim \int \frac{\chi}{r^3} v^2 \lesssim 2^{-Rj} \qquad \Rightarrow \|\nabla v\|_2 < \infty$$

$$\cancel{2\tau^{\frac12} i(\partial_r - i\tau^{\frac12}) \approx -(\partial_r - i\tau^{\frac12})^2 - r^{-2}\partial_\theta^2 - A}$$

$$\tau \|v\|_2^2 = \langle (\tau - A)v, v\rangle + \|\nabla v\|^2 < \infty$$

$$\Rightarrow v \in L^2 \quad \square$$

Outgoing cond:

(3) $$Q = b(r)(\partial_r - i\tau^{\frac12}) \qquad b(r) \approx \begin{cases} 1 & r > 2^j \\ (2^j r)^{-\delta}, & r < 2^j \end{cases}$$

$$2\,\mathrm{Re}\langle Qu, (A - \tau - i\varepsilon)u\rangle$$

$$= \langle [A, Q]u, u\rangle + \varepsilon\langle iQu, u\rangle$$

$$- \oplus\, \partial_\theta^2 - b'(r)(\partial_r - i\tau^{\frac12})^2 + \underbrace{b'(A-\tau)}_{c-S} + \text{errors}$$

$$2\tau^{\frac12} i(\partial_r - i\tau^{\frac12}) \approx \underbrace{-(\partial_r - i\tau^{\frac12})^2}_{\oplus} - \underline{r^{-2}\partial_\theta^2} - A$$

$$2\tau^{\frac12} i\langle Qu, u\rangle \gtrsim -\langle bAu, u\rangle$$

$$\|r^{-\frac12}(\partial_r - i\tau^{\frac12})u\|^2_{L^2(|x| \approx 2^j)} \lesssim \sum_k \underbrace{2^{-\delta(k-j)^-}}_{\downarrow 0} \|\langle r\rangle^{\frac12}(A - \tau - i\varepsilon)u\|^2_{L^2(|x| < 2R)} + \|\langle r\rangle^{-\frac12}(u, \nabla u)\|_{L^2}$$

$$Q = b(R)\left(\frac{x^i a^{ij}}{R}\partial_j - i\tau^{\frac12}\right); \quad R^2 = x_i a^{ij} x_j$$