$$\mathrm{div}(a(x)\nabla f) + \lambda^2 \rho(x) f(x) = 0$$

a, ρ real $\qquad a$ strict pos def

$\rho > 0$

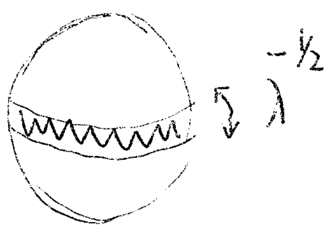
$\lambda > 0$ frequency

$\varphi_{\lambda_j} \qquad \lambda_j \to \infty$

Spectral clusters $\qquad f = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j \varphi_{\lambda_j}$

$$\frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \leq C \lambda^{\gamma(p)}$$

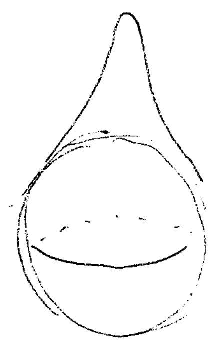
1)



Highest weight

occupying smallest volume

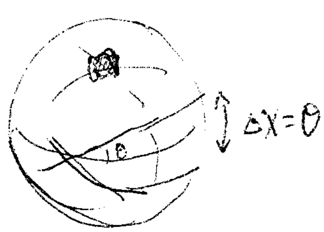
2)

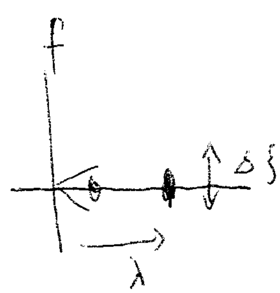


$(1+\lambda \sin\varphi)^{-1/2}$

zonal eigenfunction

Highest peak value





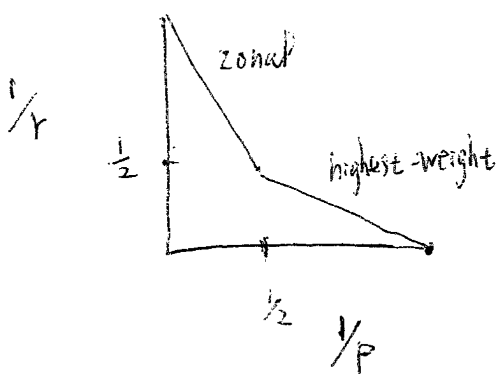
$\Delta\xi = \lambda\theta$

$\Delta x \Delta\xi \geq 1, \quad \lambda\theta^2 \geq 1$

$\theta = \lambda^{-1/2}$

1) $\dfrac{\|f\|_{L^p}}{\|f\|_{L^2}} = \lambda^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}$

2) $\dfrac{\|f\|_{L^p}}{\|f\|_{L^2}} = \lambda^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}$



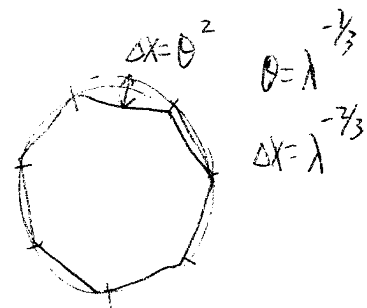
$p=6 \quad \sigma(6)=1/8$

Sogge estimates hold

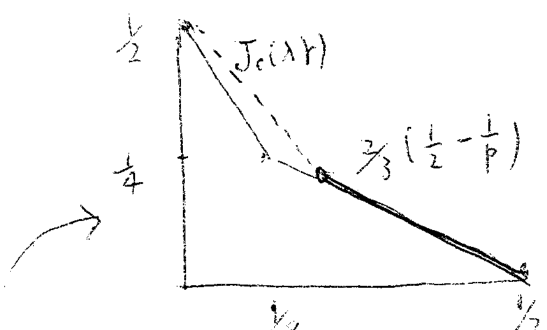
a, ρ smooth
no boundary

Grieser [1990] Sogge's estimates fail on $\{|x|\le 1\}$

$$\Delta f + \lambda^2 f = 0 \ , \quad f=0 \ , \quad |x|=1$$



$$e^{in\theta} J_n(c_0 r) \qquad \lambda \approx n$$



(S-Sogge) (2004)

estimates hold any 2-d manifold with boundary

Manifold w/ boundary $\Rightarrow$ manifold w/o bdry w/ Lipschitz metric

$(M, \partial M) \Rightarrow \tilde{M} = M \cup M$

Disc $\quad X_2 = 1-r \quad X_1 = \theta$

$$\frac{\partial^2}{\partial X_2^2} + (1+|X_2|)\frac{\partial^2}{\partial X_1^2}$$

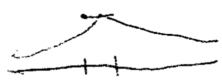
If a, ρ Lipschitz how close we get to this for spectral cluster estimate

Step 1 estimates hold if $2 \le p \le 6$ and $p = \infty$

$$\|f\|_{L^6(Q_{\lambda^{-1/3}})} \le C \lambda^{\gamma(6)} \|f\|_{energy(Q^*_{\lambda^{-1/3}})}$$

Questions:

$$d a d f + \lambda^2 \rho f = \lambda g \qquad \|f\|_{L^2} + \|g\|_{L^2} \le 1$$



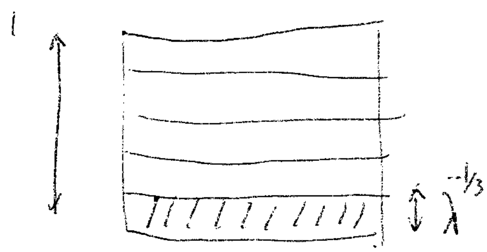
$$\lambda_j^2 - \lambda^2 = \lambda$$

$\hat{f}(\xi)$ localized to $|\xi_2| \ll \lambda$

$$\|f\|_{energy} = \|f\|_{L^\infty_{x_1} L^2_{x_2}} + \lambda^{-1} \|(d a d + \lambda^2 \rho) f\|_{L^1_{x_2} L^2_{x_2}}$$

A. Lip metric is well approximated by C^2 metric on scale $\lambda^{-1/3}$

Find a_λ : after scaling by $\lambda^{-1/3}$, $a_\lambda \to C^2$

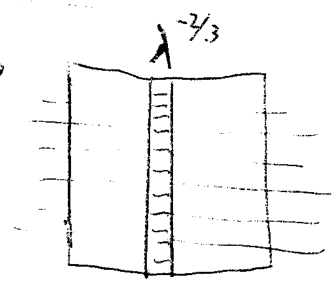


add up $\lambda^{1/3}$ pieces

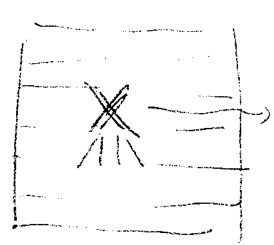
$$\lambda^{\frac{1}{3} \cdot \frac{1}{6}}$$

$$\|f\|_{L^6(M)} \le C \lambda^{\frac{1}{6} + \frac{1}{6} \cdot \frac{1}{3}} \|f\|_{L^2}$$

$p = 6$



$p > 6$ worst case



(Koch-S-Tataru)

Suppose $\|f\|_{L^p(Q)} \le C\lambda^{r(p)}\|f\|_{energy}$ holds all Q diameter δ^2

$$\not\Rightarrow \quad \|f\|_{L^{p+2}(\bar{Q})} \le C\lambda^{r(p+2)}\|f\|_{eng(\bar{Q})}$$

$\bar{Q}$ diameter δ

$p=6$, no loss scale $\lambda^{-1/3}$, $\|f\|_{L^6(M)} \le C\lambda^{r(6)+\frac{1}{6\cdot 3}}\|f\|_{L^2}$

$p=8$, log loss scale $\lambda^{-1/6}$ $\|f\|_{L^8(M)} \le C^r\lambda^{r(8)+\frac{1}{8\cdot 6}}\|f\|_{L^2}$

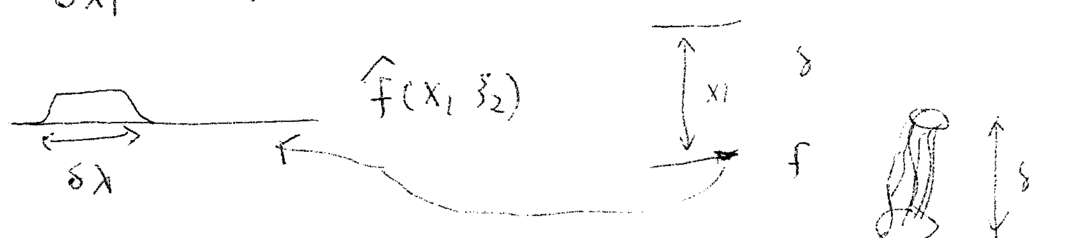
$p=10$ $\lambda^{-1/12}$ loss $\frac{1}{3p}\frac{1}{2^{\frac{p-6}{2}}}$

$$D_{x_1}^2 f = P(x,D,\lambda)f$$

Classically: $\frac{\partial \xi_1}{\partial x_1} = -\partial_{x_2}H(x,\xi,\lambda) \in L^\infty$ $\leftarrow H = \sqrt{P(x,\xi,\lambda)}$

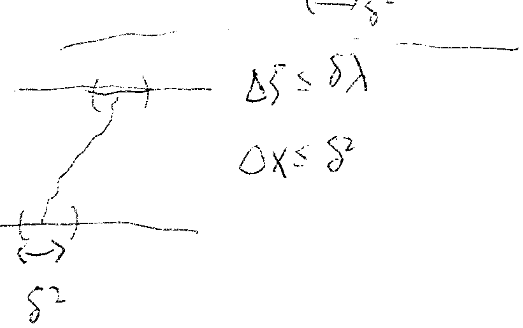
$$\frac{\partial x_2}{\partial x_1} = -\partial_{\xi_2}H(x,\xi,\lambda)$$

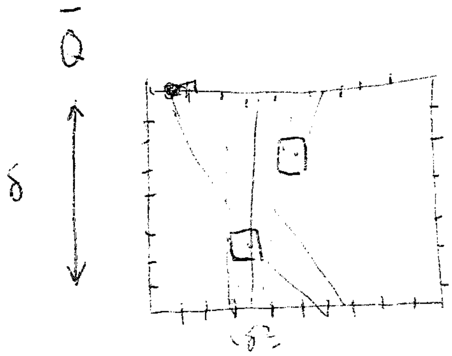
$\frac{\partial \xi_1}{\partial x_1} \le \lambda$



For ξ_2, $\partial_{\xi_2}H$ Lipschitz in X

$\frac{\partial x_2}{\partial x_1}$ well-defined





$$f = \sum f_T$$

$$\sum \|f_T\|^2_{eng} \lesssim \|f\|^2_{eng}$$

Overlap estimates

2 cubes Q (diameter δ^2)

$$\# \; T \text{ in common } \lesssim \frac{\delta}{|x_1 - y_1|}$$

$$N \text{ tubes} \quad n_Q = \#T : T \cap Q \neq \varphi$$

$$\|n_Q\|_{\ell^2_{x_1} \ell^\infty_{x_2}} \leq N^{\frac{1}{2}} \delta^{\frac{1}{2}} |\log \delta|^{\frac{1}{2}}$$

$$\Rightarrow \|n_Q\|_{\ell^2} \leq N^{1-\frac{1}{q}} \delta^{-\frac{1}{q}} |\log \delta|^{-\frac{1}{2}}$$

$$\|n_Q\|_{\ell^\infty_{x_1} \ell^1_{x_2}} \leq N$$

$$f = \sum_{N \text{ tubes}} f_T \quad , \quad \|f_T\|_{energy} = 1$$

$$\|f\|_{L^p(Q)} \leq C \lambda^{\gamma(p)} n_Q^{1/2}$$

$$\|f\|_{L^\infty(Q)} \leq C \lambda^{1/2} \delta^{1/2} n_Q$$

$$\|f\|_{L^p(\bar{Q})} \leq C \lambda^{\gamma(p)} N^{\frac{1}{2} - \frac{1}{p}} \delta^{-\frac{1}{p}} |\log \delta|^{\frac{1}{p}}$$

$$\|f\|_{L^{p+2}(\bar{Q})} \leq C r^{\gamma(p+2)} \underset{\|f\|_{energy}}{\underset{\sim}{N}}^{\frac{1}{2}} |\log r|$$

$$\|f\|_{L^\infty(\bar{Q})} \leq C \lambda^{\gamma(\infty)} N \delta^{\frac{1}{2}}$$