$C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, smooth

$\Omega, \Omega^*$

A1. $x \in \Omega, y \in \Omega^*, p, q \in \mathbb{R}^n$

$\Rightarrow \exists Y, X$ unique such that

$C_x (x, Y) = p, \quad C_y (X, y) = q$

A2. $| \det C_{x,y} | \geq C_0, \quad C_0 > 0$

$$\det \left[ D^2 u - D_x^2 C (x, Y, Du) \right] = | \det C_{x,y} | \begin{pmatrix} f(x) \\ A(x, Du) \end{pmatrix}$$

$$B(x, Du)$$

Mass balance condition

$$\int_{\Omega} f = \int_{\Omega^*} g, \quad f, g > 0$$

Second boundary condition

$$T (\Omega) : \gamma (\cdot, Du) (\Omega) = \Omega^*, \quad \text{diffeomorphism}$$

$$\max \left( \int_{\Omega} f + C (x, TX) dx \right)$$

Theorem: Conditions

$$\frac{\partial}{\partial y_1} C_{1k} \frac{\partial}{\partial y_2} C_{2k} \begin{cases} C(x, y) = x \\forall \Omega, \Omega^* \text{ unit convex} \\ C^{k, k} \text{ matrix of } C_{x, y} \end{cases}$$
\[ A^3 \quad \left( D_{\alpha} \eta \right) \eta_{ij} \eta_{kl} \eta_{m} \geq C_{0} \left| \eta \right|^2 \quad \forall \eta \in \mathbb{R}^n, \quad C_0 > 0 \]

\[ A^3 \quad (\geq 0) \]

Domains:

(\text{unit})

\[ \Omega \text{ is } C \text{ convex w.r.t. } \Omega^* \]

\[ \iff \quad C_y (\cdot, y)(\Omega) \text{ Convex } \quad \forall y \in \Omega^* \]

\[ \Omega^* \text{ is } C^* \text{ convex, } \quad C^*(x, y) = C(y, x) \]

Theorem:

\[ f, g \text{ smooth } \implies f, g > 0 \]

\[ \Omega, \Omega^* \text{ unif } C, C^* \text{ convex w.r.t. each other} \]

\[ C \text{ satisfies } A_1, A_2, A_3 \quad \text{(w)} \]

\[ \implies \exists \text{ globally smooth elliptic soln of second b.v. problem (unique up to constant)} \]

\[ \text{Cor: problem has smooth diffeomorphism} \]

Local Theorem:

\[ \Omega, \Omega^*, C, C^* \text{ w.r.t. each other} \]

\[ C, A_3 \implies \text{ problem has locally smooth diffeomorphism} \]
proof: from a priori method of continuity

Construct a smooth function $u_1$ and approximately satisfies $u \circ T_{y_0} \approx \Omega^+$

Need under conditions of Thm

ellipticity $\Rightarrow$ C. Convexity

i.e. $u(x) - u(x_0) \geq c(x, y_0) - c(x_0, y_0)$

for $y_0 = y(x_0, Du(x_0)) \forall x \in \Omega$

in general non-smooth u want same $y_0$

set of such $y_0$ is called c-convex mapping of u

function of form $f(x) = c(x, y_0) + \text{const}$ are called Cauchy functions.

\[ u_1 = \sup \{ h \mid h \leq u_0 \text{ in } \Omega^+ \} \]

Need to show $u_1$ extends $u_0$

smooth in $\Omega^\circ - \Omega^+$

need $A_0$ modify $u_1$ so that it is elliptic in $\Omega^\circ - \Omega$
finally we need to smooth it.

\[ y_0, \quad y = \gamma(x_0, C(x_0, y_0) + t\eta) \]
\[ \Rightarrow c(x, y) - c(x_0, y) < c(x, y_0) - c(x_0, y_0) \]
on \( \Omega - \{x_0\} \)

needs \( A; w + \text{unit} \circ \text{convex w.r.t. } y_0 \)

\[ \Rightarrow \text{extension} \]