$C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, smooth $\quad \Omega$, Ω^*

A1. $x \in \Omega$, $y \in \Omega^*$, $p, q \in \mathbb{R}^n$

$\Rightarrow \exists\ Y, X$ unique such that

$C_x(x, Y) = p, \quad C_y(X, y) = q$

A2 $\quad |\det C_{x,y}| \geq C_0$, $C_0 > 0$

$$\det\left[D^2u - \underbrace{D_x^2 C(\cdot, Y(\cdot, Du))}_{A(x, Du)}\right] = \underbrace{|\det C_{x,y}| \frac{f(x)}{g(y)}}_{B(x, Du)}$$

Mass balance condition

$$\int_{\Omega} f = \int_{\Omega^*} g \quad , \quad f, g > 0$$

Second boundary condition

$T(\Omega) := Y(\cdot, Du)(\Omega) = \Omega^*$, diffeomorphism

$$\max \int_{\Omega} f\, c(x, Tx)\, dx$$

<u>Theorem</u>:

<u>Conditions</u> $\quad \frac{\partial}{\partial p_k} = C^{kl} \frac{\partial}{\partial y_l}$ $\quad \begin{cases} C(x,y) = x\ y \\ \Omega, \Omega^* \text{ unif convex} \end{cases}$

$[C^{k,l}]$ matrix of $C_{x,y}$

A3 $\left(D_{p_k p_l} C_{ij}\right) \xi_i \xi_j \eta_k \eta_l \geq C_0 |\xi|^2 |\eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^n, \; C_0 > 0$

A3W $\quad (\geq 0)$

Domains:

Ω is (unif) c convex w.r.t. Ω^*

$\Leftrightarrow C_y(\cdot, y)(\Omega)$ (unif) convex $\forall \; y \in \Omega^*$

Ω^* is c^* convex, $\; c^*(x,y) = c(y,x)$

Theorem: f, g smooth $\inf f, g > 0$

Ω, Ω^* unif c, c^* convex w.r.t. each other

c satisfies A1, A2, A3w

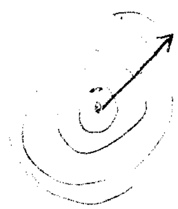
$\Rightarrow \exists$ globally smooth elliptic soln of second b.v. problem (unique up to constant)

Cor: problem has smooth diffeomorphism

local Theorem: Ω, Ω^*, c, c^* w.r.t each other

c, A3 $\Rightarrow$ o t problem has locally smooth diffeomorphism

proof: from a priori
method of continuity



Construct a smooth function for elliptic u_2
and approximately satisfies b.c. $T_{u_1}(\Omega) \sim \Omega^*$

Need under conditions of Thm

ellipticity $\Rightarrow$ C. convexity

i.e. $u(x) - u(x_0) \geq C(x, y_0) - C(x_0, y_0)$

for $y_0 = Y(x_0, Du(x_0))$ $\forall x \in \Omega$

in general non-smooth u want same y_0

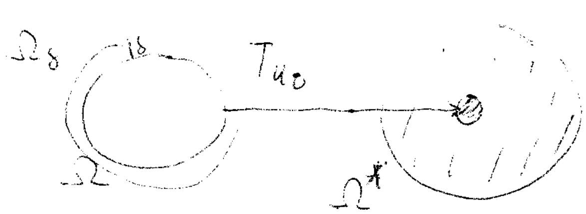
Set of such y_0 is called c-convex mapping of u

function of form $h(x) = c(x, y_0) + \text{const}$ are called

Cauchy functions.

$u_1 = \sup\{ h \mid h \le u_0 \text{ in } \Omega \}$

$y_0 \in \Omega^*$

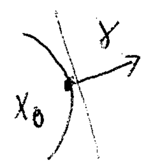


Need to show $\begin{cases} u_1 \text{ extends } u_0 \\ \text{smooth in } \Omega_\delta - \partial\Omega \end{cases}$

need to modify u_1 so that it is elliptic in $\Omega_\delta - \overline{\Omega}$

finally we need to smooth it. ~~[illegible]~~

Lemma:



y_0, $\quad y = Y(x_0, C_x(x_0, y_0) + t\gamma)$

$\Rightarrow \quad c(x,y) - c(x_0, y) < c(x, y_0) - c(x_0, y_0)$

on $\bar{\Omega} - \{x_0\}$

needs A; w + unif - convex w.r.t. y_0

$\Rightarrow$ extension