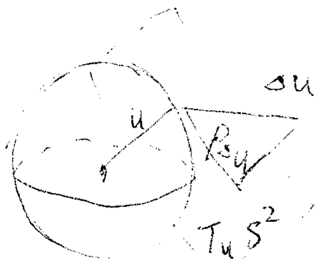
$u : \mathbb{R}^n \times \mathbb{R} \to S^2 \subset \mathbb{R}^3$

A Schrödinger map

$$(1) \begin{cases} u_t = u \times \Delta u \\ u(0) = u_0 \end{cases} \qquad u_t = JP\Delta u$$

$$i V_t = \Delta V \qquad V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$$



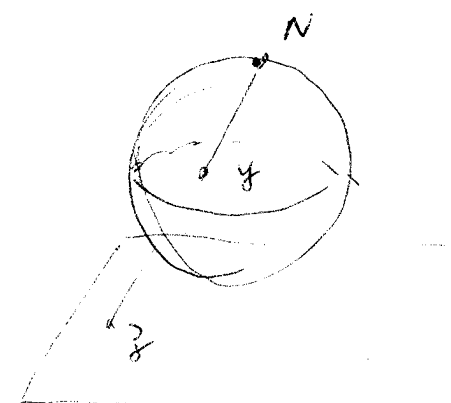
Harmonic Map

$$\phi : \mathbb{R}^n \to (M, g) \quad , \quad L = \int |\nabla \phi|_g^2 \, dx$$

$$\Delta \phi^i = \Gamma^i_{jk} \partial_\alpha \phi^j \partial_\alpha \phi^k$$

$$\phi : \mathbb{R}^n \times \mathbb{R} \to (M, g)$$

$$L = \int -|\phi_t|_g^2 + |\nabla \phi|_g^2 \, dx$$



Wave Maps

$$\Box \phi^i = \Gamma^i_{jk}(\phi) \partial_\alpha \phi^j \partial^\alpha \phi_k$$

$$z = \frac{u_1 + i u_2}{1 + u_3} \quad , \quad i z_t - \Delta z = \frac{2 \bar{z} (\nabla z)^2}{1 + |z|^2}$$

$$S^2 \setminus N \sim (\mathbb{C}, g \, dz \, d\bar{z}) \quad , \quad g = (1 + |z|^2)^{-2}$$

$$z : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$$

$$L = \int_{\mathbb{R}^n} \frac{|\nabla z|^2}{(1+|z|^2)^2}\, dx$$

$$\sum_{j=1}^{n} \Big(\underbrace{\frac{\partial}{\partial x_j} + \frac{2\bar{z}}{1+|z|^2} \frac{\partial z}{\partial x_j}}_{\nabla_j} \Big) \frac{\partial z}{\partial x_j} = 0$$

$$\boxed{i z_t = \Delta z + \frac{z\bar{z}}{1+|z|^2} (\nabla z)^2} \quad \text{SM}$$

$u_0 \in H^s_Q$

$u_0 - Q \in H^s$

$$(2) \begin{cases} i z_t - \Delta z = \frac{z\bar{z}}{1+|z|^2} (\nabla z)^2 \\ z(0) = z_0 \in H^s \end{cases}$$

Kenig, Pollack, Staffilani, Toro

$u_0 \in H^s(\mathbb{R}^n \times \mathbb{R}, K) \rightsquigarrow$ compact Kähler manifold

<u>LWP</u>

Scaling $\quad u_\lambda = u(\lambda x, \lambda^2 t) \quad \rightarrow \quad s_c = \frac{n}{2}$

Energy $\quad \|\nabla u\|_{L^2} = \text{constant}$

Expectations:

1) $u_0 \in H^s$, $s > s_c$, LWP

2) scaling, small initial data $\quad$ GW ?

large initial data

GWP 1d $\quad (n = 1,\ s_c = \frac{1}{2}, \rightarrow \dot{H}^{\frac{1}{2}}\ \ H^1)$

Wave Map 1) Klainerman specific structure - null condition

$(u_0, u_1) \in H^s \times H^{s-1}$, $s > s_c$ LWP

2) $s = \frac{n}{2}$

Tataru $(u_0, u_1) \in B^{n/2}_{2,1} \subset L^\infty \times B^{n/2-1}_{2,1}$ GWP $n \geq 4$, $n = 2, 3$

3) Tao smooth initial data with small $H^{n/2} \times H^{n/2-1}$ norm global in time.

$i u_t - \Delta u = f$

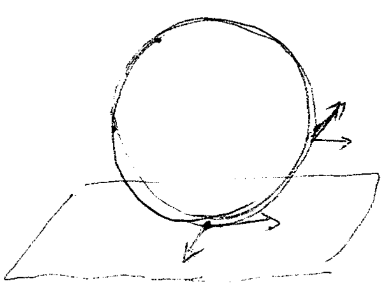
$u(0) = u_0$

$$\sup_\alpha \|\nabla^{1/2} u\|_{L^2(Q_\alpha \times \mathbb{R})} \leq \|u_0\|_{L^2} + \sum_\alpha \|\nabla^{-1/2} f\|_{L^1(Q_\alpha \times \mathbb{R})}$$

Kenig, Ponce, Vega, Chihara

$$i u_t - \Delta u = P(u, \bar{u}, \nabla u, \nabla \bar{u})$$
$$= B(\nabla u, \nabla u)$$

$s = s_c + 1$



Hashimoto transformation

$$q = e^{i\psi} \frac{2x}{1+|z|^2}$$

$$i q_t - \Delta q = \frac{1}{2} q |q|^2$$

$q_0 \in L^2 \Rightarrow$ GWP

$\|q(t)\|_{L^2} = ct.$

$n=2$ MSM system

local existence for $u_0 \in H^{\frac{3}{2}+\varepsilon}$

Kato Kenig, Nahmod

uniqueness $u_0 \in H^{7/3+\varepsilon}$ Kato, Koch

Th (B) $s > s_c = \frac{n}{2}$, $\| z_0 \|_{H^s} < \delta$, then (2) has

a unique sol in $C_t H^s \cap X^s$

Thm (Ionescu, Kenig) $s > s_c + \frac{1}{2}$, $n \geq 2$

$$i z_t - \Delta z = (\nabla z)^2 \quad , \quad B(u,v) = \nabla u \nabla v$$

$X^{s,b}$ space $\quad \langle \xi \rangle^s \langle \tau - \xi^2 \rangle^b \hat{f} \in L^2_{\tau,\xi}$

$b > \frac{1}{2}$, $X^{s,b} \subset C_t H^s$

$X^{s,\frac{1}{2},1}$

$$\lambda < \mu$$

$$\| e^{-it\Delta} u_{0,\lambda} \cdot e^{-it\Delta} v_{0,\mu} \|_{L^2} \lesssim \lambda^{\frac{n-1}{2}} \mu^{-\frac{1}{2}} \| u_{0,\lambda} \|_{L^2} \| v_{0,\mu} \|_{L^2}$$

$$2 \nabla u \cdot \nabla v = \underbrace{(i\partial_t - \Delta)}_{(\tau - \xi^2)} uv + u(i\partial_t - \Delta)v + (i\partial_t - \Delta)(uv)$$

$X^{0,-\frac{1}{2},1}$

$$\| e^{-it\Delta} u_{0,\lambda} e^{-it\Delta} v_{0,\mu} \|_{X^{0,-\frac{1}{2},1}} \le \lambda^{\frac{n}{2}} \| u_{0,\lambda} \|_{L^2} \| v_{0,\mu} \|_{L^2}$$

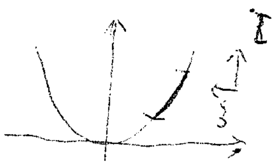
$$\| B(u_\lambda, u_\mu) \|_{X^{0,-\frac{1}{2},1}} \le \lambda^{\frac{n}{2}} \log \mu \, \| u_\lambda \|_{X^{0,\frac{1}{2},1}} \| v_\mu \|_{X^{0,\frac{1}{2},1}}$$

Ionescu, Kenig. lateral energy estimates

$$\begin{cases} iu_t - \Delta u = f \\ u(0) = u_0 \end{cases} \qquad \hat{u}_0 , \hat{f} \text{ are supported at fre } \lambda \quad |\xi_1| \geq \tfrac{1}{2}|\xi|$$

$$\|u\|_{L^\infty_{x_1} L^2_{t,x'}} \lesssim \lambda^{-1/2} \|u_0\|_{L^2} + \lambda^{-1} \|f\|_{L^1_{x} L^2_{t,x'}}$$

$\lambda = 1$ $\quad \|e^{-it\Delta} u_0\|_{L^\infty_{x_1} L^2_{t,x'}} \lesssim \|u_0\|_{L^2}$



$L^\infty_t L^2_x$

$L^\infty_x L^2_t$

$L^\infty_{x_1} \; L^2_{t,x'}$ $\quad \underbrace{L^2_{x_1} L^\infty_{t,x'}}_{L^1}$ $\quad \|e^{-it\Delta} u_0\|_{L^2_x L^\infty_{t,x'}} \lesssim \lambda^{\frac{n-1}{2}} \|u_0\|_{L^2}$

$\frac{1}{\langle t,x \rangle^{1/2}}$ $\quad \boxed{n=2}$

L^∞ $\quad n^{n/2}$ $\quad u_0 \in B^{n/2}_{2,1}$ small $\quad$ GWP

Thm (B, Ionescu, Kenig) $\quad u_0 \in H^\infty_Q$, $\|u_0\|_{\dot{H}^{n/2}} \leq \varepsilon(n)$

Global solutions $\quad u \in C_t H^s(\mathbb{R} \times \mathbb{R}^n)$

$n \geq 4$