Local Smoothing:

$$-\Delta = -\sum \partial_{x_j}^2$$

$\mathbb{R}^2$

$$e^{it\Delta} : H^s \to H^s$$

~87 Constantin-Saut, Sjölin, Vega

$\boxed{\text{LS}}$ $$\int_0^T \| \chi e^{-it\Delta} u \|_{H^{1/2}}^2 \, dt \leq C \| u \|_{L^2}^2$$

$$\chi \in C_c^\infty$$

Follows from resolvent estimates:

$\boxed{\text{RE}}$ $$\chi(-\Delta - \lambda \pm i0)^{-1} \chi = \mathcal{O}_{L^2 \to L^2}\left(\frac{1}{\sqrt{\lambda}}\right)$$

<u>Proof</u> $$Tu(t,x) = \mathbb{1}_{[0,T]}(t)\, \chi(x) \left(e^{-it\Delta} u\right)(x)$$

$\boxed{\text{LS}} \iff T : L^2 \to L_t^2 H_x^{1/2}$

$$T^* v(x) = \int_0^T e^{is\Delta} \left(\chi(\cdot)\, v(\cdot, s) \right)(x)\, ds$$

$\boxed{\text{LS}} \iff TT^* : L_t^2 H_x^{-1/2} \to L_t^2 H_x^{1/2}$

$$TT^* = L_+ + L_- \qquad L_+ v(t,x) = \mathbb{1}_{[0,T]}(t) \int_0^t \chi e^{-i(t-s)\Delta} \chi v(s,\cdot)(x)\, ds$$

$$L_+ = \mathbb{1}_{[0,T]}(t) \left(\mathbb{1}_{[0,T]}(\cdot)\, \chi e^{-i\cdot\Delta} \chi \right) * \left(v(\cdot, \cdot_x) \right)(t)$$

Plancherel $$\| L_+ v \|_{L_t^2 H_x^{1/2}}^2 \leq C \int \| \widehat{L_+ v}(\lambda, \cdot) \|_{H_x^{1/2}}^2$$

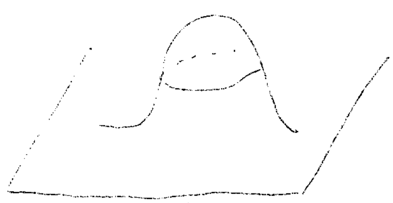
$\boxed{RE} \iff \chi(-\Delta - \lambda \pm i0)^{-1}\chi = \cancel{\mathcal{O}_{H^{-\frac{1}{2}} \to H^{\frac{1}{2}}}(1)} \quad \mathcal{O}_{L^2 \to L^2}\left(\frac{1}{\sqrt{\lambda}}\right)$

$\lambda > 0$

$$\|L_t v\|^2_{L^2_t H^{\frac{1}{2}}_x} \le C \int \|\widehat{L_t v}(\lambda, \cdot)\|^2_{H^{\frac{1}{2}}_x} \, d\lambda$$

$$\le C \int \|\chi(-\Delta_x - \lambda - i0)^{-1}\chi \, \hat{v}(\lambda, \cdot)\|^2_{H^{\frac{1}{2}}_x} \, d\lambda$$

$\boxed{RE}$ holds for nontrapping metrics i.e. $K = \emptyset$

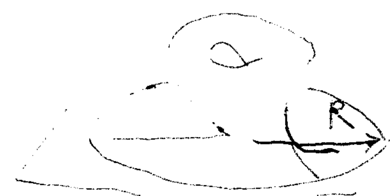


$\mathbb{R}^2, g$

$K \subset S^*X$

... Vasy-Z, Vodev 2000

what if $K \neq \emptyset$



$\mathbb{R}^2, g$

$\operatorname{supp}\chi \cap B(0, R) = \emptyset$

$\boxed{RE}$ <u>holds</u> Burq, Vodev 2000

Doi 1997, $K \neq \emptyset \Rightarrow \boxed{LS}_{1/2}$ fails

Burq 24004, Ikawa 1988 small and far apart $\alpha < \frac{1}{2}$

Christiansen 2006, $K = 1$ hyp orbit

Theorem (Nonnenmacher–Z, 2009)
simple case

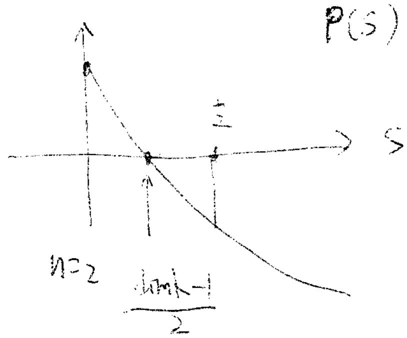
$$X = \begin{cases} \text{neg curv} & \text{cpct} \\ \mathbb{R}^2 & \text{outside cpct} \end{cases}$$

$$\dim_H K < 2 \;\Rightarrow\; \chi(-\Delta - \lambda \pm i0)^{-1}\chi = \mathcal{O}_{L^2 \to L^2}\left(\frac{\ln \lambda}{\sqrt{\lambda}}\right)$$

$$\Rightarrow \boxed{LS}_{1/2 - \varepsilon}$$

* K hypebolic

* $n > 2$ cond in terms of "pressure" $P(\tfrac{1}{2}) < 0$

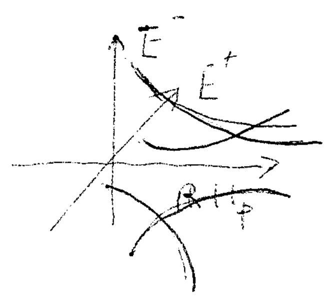


$\lambda < -P_E(\tfrac{1}{2})$

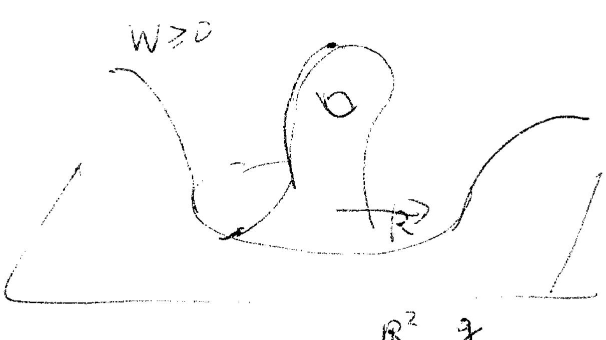
$$P = -h^2 \Delta_g + V \qquad \boxed{RE}_{\log} \iff \chi(P - E \pm i0)^{-1}\chi = \mathcal{O}_{L^2 \to L^2}\left(\frac{\log \frac{1}{h}}{h}\right)$$

$p = |\xi|_g^2 + V(x)$

$\Phi_t = \exp t H_p \quad , \quad \Phi_t|_{K_E}$ <u>Anosov</u>



$p^{-1}(E) \;,\; dp \neq 0$



$$P \rightsquigarrow P_W = P - iW$$

$\boxed{PW}$ $\quad (P - iW - E)^{-1} = \mathcal{O}_{L^2\to L^2}\left(\frac{\log\frac{1}{h}}{h}\right), \quad E>0$

- $V(t) = \exp\left(-\frac{it}{h}(P-iW)\right)\chi^w(x,hD)$

$\psi = 1, \quad |\xi|_g^2 + V = E, \quad \psi = 0$ away

$$\|V(t)\|_{L^2\to L^2} \le \begin{cases} 1 + \mathcal{O}(h), & t\ge 0 \\ C_M h^{-\frac{n}{2}} e^{-\lambda t} & 0\le t\le M\log\frac{1}{h}, \quad \forall M \\ C_M h^{\frac{M}{c}}, & t\ge M\log\frac{1}{h} \end{cases}$$

$\boxed{UT}$

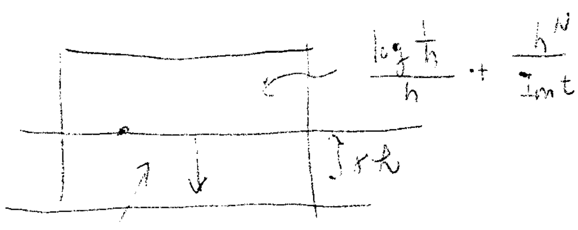
$\boxed{UT} \Rightarrow \boxed{PW}$

Proof $\quad (P - iW - z)^{-1}\psi^w = \frac{1}{h}\int_0^\infty U(t)\, e^{itz}\, dt$

$\operatorname{Im} z > 0$

$z \sim E$

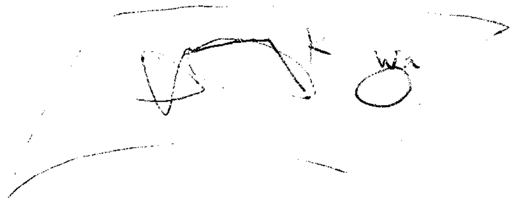
$$= \mathcal{O}_{L^2\to L^2}\left[\frac{C\log\frac{1}{h}}{h} + \frac{h^N}{\operatorname{Im} z}\right]$$



$(P - iW - z)^{-1} = \mathcal{O}_{L^2\to L^2}\left(e^{Ch^{-n-\varepsilon}}\right) \xRightarrow[\text{Tang-Z 198}]{} \boxed{PW}$

$$\|U(t)\|_{L^2\to L^2} \le C_M\, h^{-\frac{n}{2}} e^{-\lambda t} \qquad 0\le t\le M\log\frac{1}{h}$$

Anantharaman–Nonnenmacher 2006

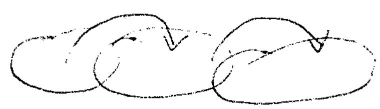


$$\sum \Pi_a = \psi^w(x,hD) + \mathcal{O}(h^\infty)$$

$$WF_\hbar(\Pi_a)\subset W_a$$

$$e^{-iNt_0 P_w/h}\,\psi^w = \sum_{\alpha\in A_1^N \cap A_N} U_{\alpha_N}\cdots U_{\alpha_1}\ ,\quad U_\alpha = e^{-\frac{it_0 P_w}{\hbar}}\,\Pi_\alpha$$

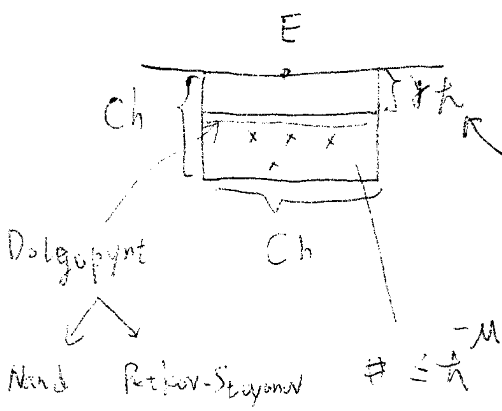
Most terms are negligible



$W\ge 1$ on $\pi(W_\alpha)$, $U_\alpha = \mathcal{O}(\hbar^\infty)$

$$\|U_{\alpha_N}\cdots U_{\alpha_0}\| = \hbar^{-\frac{n}{2}}(1+\varepsilon)^N \times \prod_i^N \inf_{\rho\in W_{\alpha_j}\cap K_E^s} \det\left(d\Phi^{t_0}\big|_{E_\rho^+}\right)^{-\frac12}$$

$$N\le M\log\frac{1}{h}$$



$\gamma < -P_E(\frac12)$

$\dim_H K = 2\mu+1$

$\mu<\frac12 \Rightarrow \gamma>0$ (NZ)

Sjöstrand – Z