Global existence for elastic waves in an exterior domain

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I. Introduction

Assume we have an elastic material filling space. Let

$$\varphi(t,x) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$$

be an orientation preserving diffeomorphism taking material points $x \in \mathbb{R}^3$ in reference configuration to position $\varphi(t,x) \in \mathbb{R}^3$.

Define $F^i_j \equiv \frac{\partial \varphi^i}{\partial x^j}$ the deformation gradient
φ orientation preserving $\Rightarrow \ \det F > 0$.
incompressible materials $\Leftrightarrow \ \det F = 1$.

Constitutive assumptions:

Homogeneous Cauchy Elastic Material There exists a stress tensor $S(t,x)$ called the Piola-Kirchhoff stress tensor in the reference configuration which depends only on F.

Hyperelastic There exists $W(F)$ the stored energy function such that $S^i_j(F) = \frac{\partial W}{\partial F^i_j}$.

Isotropic and Frame Indifferent $W(FQ) = W(QF) = W(F)$ for all proper orthogonal matrices Q. Equivalently this means that W depends on F only through the principle invariants of the strain tensor $F^{\mathrm{T}}F$.

- Equations of motion:

$$\frac{\partial^2 \varphi^i}{\partial t^2} - \frac{\partial}{\partial x_\ell}\left(\frac{\partial W}{\partial F^i_\ell}(\nabla\varphi)\right) = 0$$

- Consider small displacements from the identity:

$$\varphi(t,x) = x + u(t,x).$$

- Truncate to quadratic nonlinearity:

$$Lu = \partial_t^2 u - Au = N(u,u)$$

$$(Au)^i = A^{ij}_{\ell m}\partial_\ell\partial_m u^j \qquad N(u,v) = B^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j \partial_n v^k)$$

$$A^{ij}_{\ell m} = \frac{\partial^2 W}{\partial F^i_\ell \partial F^j_m}(I) \qquad B^{ijk}_{\ell mn} = \frac{\partial^3 W}{\partial F^i_\ell \partial F^j_m \partial F^k_n}(I)$$

- By definition we have nice symmetry properties:

$$A^{ij}_{\ell m} = A^{ji}_{m\ell}, \quad \text{and} \quad B^{ijk}_{\ell mn} = B^{jik}_{m\ell n} = B^{ikj}_{\ell nm}.$$

- If u is small we can assume that the linear equations will be hyperbolic.

- Linear Equations (for isotropic solid):

$$Lu = \partial_t^2 u - c_2^2 \triangle u - (c_1^2 - c_2^2)\nabla(\nabla \cdot u) = 0.$$

- Ellipticity condition: $c_1^2 > \frac{4}{3}c_2^2 > 0$, equivalent to the *Legendre-Hadamard* condition and implies that the linear operator is hyperbolic, or the symbol of A is positive definite.

- Helmholtz - Hodge decomposition: Projecting onto curl free or divergence free subspaces will split the system into coupled wave equations for pressure waves and shear waves.

- Wave propagation speeds:

  c_1 is the speed of the pressure waves

  c_2 is the speed of the shear waves

- Incompressible elasticity: Characterized by $\det F = 1$, formally send $c_1 \to \infty$. Morally left with a wave equation for shear waves. Can extend some existence results from compressible to incompressible elasticity via this limit.

- Fluids: The Euler equations are recovered in the formal limit $c_1 \to \infty$, and $c_2 \to 0$. Our methods require ellipticity which rules this limit out.

- Plane Waves: For each direction $\xi \in S^2$, there are two families of elementary plane wave solutions to $Lu = 0$.

$$\mathcal{W}_1(\xi) = \{\alpha\xi \exp i\beta[\langle x, \xi\rangle - c_1 t] : \alpha, \beta \in \mathbb{R}\}$$
$$\mathcal{W}_2(\xi) = \{\eta \exp i\beta[\langle x, \xi\rangle - c_2 t] : \langle \eta, \xi\rangle = 0, \beta \in \mathbb{R}\}$$

  $\mathcal{W}_1$ represents longitudinal (or pressure) waves propagating with speed c_1 and $\mathcal{W}_2$ represent transverse or shear waves propagating with speed c_2.

- Null Condition: Nonresonance condition on the quadratic portion of the nonlinearity. N is null with respect to the linear operator if

$$\langle u, N(v, w)\rangle = 0,$$

  for all resonant triples

$$(u, v, w) \in \mathcal{W}_\alpha \times \mathcal{W}_\alpha \times \mathcal{W}_\alpha, \quad \alpha = 1, 2$$

In terms of the coefficients of the nonlinearity this corresponds to

$$B^{ijk}_{\ell mn}\xi_i\xi_j\xi_k\xi_\ell\xi_m\xi_n = 0, \quad \text{for all } \xi \in S^2,$$
$$B^{ijk}_{\ell mn}\eta_i^{(1)}\eta_j^{(2)}\eta_k^{(3)}\xi_\ell\xi_m\xi_n = 0, \quad \text{for all } \xi, \eta^{(a)} \in S^2 \text{ with } \left\langle \xi, \eta^{(a)} \right\rangle = 0.$$

Due to the structure of the stored energy function for an isotropic solid the second condition above is redundant, and the null condition is the first condition.

$$\begin{cases} (Lu)^i = B^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j \partial_n u^k), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \backslash \mathcal{K}, \\ u|_{\partial\mathcal{K}} = 0, \\ u(0,\cdot) = f, \quad \partial_t u(0,\cdot) = g \end{cases} \tag{1}$$

Theorem (Metcalfe-T '07): Let $\mathcal{K}$ be a bounded, nontrapping obstacle with smooth boundary. Assume that $A^{ij}_{\ell m}$ satisfies the ellipticity condition and that $B^{ijk}_{\ell mn}$ satisfies the null condition. Suppose that $(f,g) \in C^\infty(\mathbb{R}^3 \backslash \mathcal{K})$ satisfy certain compatibility conditions to infinite order. Then, there are positive constants ε_0 and N so that for all $\varepsilon < \varepsilon_0$, if

$$\sum_{|\alpha|\le N} \| \langle x \rangle^{|\alpha|} \partial_x^\alpha f \|_2 + \sum_{|\alpha|\le N-1} \| \langle x \rangle^{|\alpha|+1} \partial_x^\alpha g \|_2 \le \varepsilon,$$

then the system (1) has a unique global solution

$$u \in C^\infty([0,\infty) \times \mathbb{R}^3 \backslash \mathcal{K}).$$

Brief History:

- Nonlinear Wave Equation in $\mathbb{R}^3$:
  - ’85 Klainerman: Vector field method for nonlinear wave equations
  - ’86 Klainerman and Christodoulou: Global existence for 3D quasilinear wave equations for small data with Null Condition
  - ’96 Klainerman-Sideris: Almost global existence for nonlinear wave equations via vector field method, **without** Lorentz boosts. Applies to elasticity

- Elasticity in $\mathbb{R}^3$:
  - ’88 John: Almost global existence for elastic waves (direct estimation of fundamental solution)
  - ’00 Sideris: Global existence for nonlinear elasticity with Null condition
  - ’84 John: Blow up for spherically symmetric elasticity if genuinely nonlinear.
  - ’98 Tahvildar-Zadeh: Blow up for sufficiently large displacements in the absence of the null condition
  - ’05 T-Sideris: Incompressible elasticity has global solutions, via the incompressible limit (namely sending $c_1 \to \infty$)

- Exterior Domain $\mathbb{R}^3 \backslash \mathcal{K}$:
  - ’02 Keel-Smith-Sogge: Global existence for quasilinear wave equation outside star-shaped obstacle
  - ’05 Metcalfe-Sogge: Global existence for quasilinear wave equation outside nontrapping obstacle
  - ’06 Metcalfe: Almost global existence for elastic waves outside of star-shaped obstacle
  - ’07 Metcalfe-T: Global existence for elastic waves outside of nontrapping obstacle.

Vector Fields:

Spatial translation ∂_j for $j = 1,2,3$

Dilation (scaling) $S = t\partial_t + r\partial_r - 1 = t\partial_t + \frac{x_j}{|x|}\partial_j - 1$ (modified to preserve null structure of equations)

Generators of simultaneous rotation $\tilde{\Omega}_\ell = \Omega_\ell I + U_\ell$, with $\Omega = x \times \nabla$, generators of spatial rotations, where U_ℓ are

$$U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Crucially:

$$[L,\partial] = [L,\tilde{\Omega}] = 0, \quad \text{and} \quad [L,S] = 2L.$$

Note: Linear Elasticity operator does not commute with the Lorentz boosts $\Omega_{0i} = t\partial_i + x_i\partial_t$

Thus define: $\Gamma = (\partial, \tilde{\Omega}, S)$ and $Z = (\partial, \tilde{\Omega})$ we need Z in order to keep track of the number of scaling vector fields that appear for the boundary estimates.

Global existence for elastic waves with null condition:

Energy:

$$E_1(u(t)) = \frac{1}{2}\int \left[|\partial_t u(t)|^2 + c_2^2|\nabla u(t)|^2 + (c_1^2 - c_2^2)(\nabla\cdot u(t))^2\right] dx,$$

and higher energies $E_k(u(t)) = \sum_{|a|\leq k-1} E_1(\Gamma^a u(t))$.

Define orthogonal projections onto radial and transverse directions:

$$P_1 u(x) = \frac{x}{r}\otimes\frac{x}{r}u(x) = \frac{x}{r}\left\langle\frac{x}{r}, u(x)\right\rangle,$$
$$P_2 u(x) = [I - P_1]u(x)) = -\frac{x}{r}\wedge\left(\frac{x}{r}\wedge u(x)\right),$$

Weighted L^2 norm:

$$\mathcal{X}_k(u(t)) = \sum_{\alpha=1}^{2}\sum_{\beta=0}^{3}\sum_{\ell=1}^{3}\sum_{|a|\leq k-2} \|\langle c_\alpha t - r\rangle P_\alpha \partial_\beta \partial_\ell \Gamma^a u(t)\|_{L^2}.$$

Using the scaling vector field and the rotations one can obtain bounds on this type of weighted norm in terms of the energy of the form:

$$\mathcal{X}_k(u(t)) \le C E_k^{1/2}(u(t)) + C \sum_{|a|\le\mu-2} t\|L\Gamma^a u(t)\|_{L^2}.$$

This has been generalized for certain isotropic symmetric hyperbolic systems (Sideris-T '05).

With Sobolev estimates these give weighted L^∞ estimates such as

$$\begin{aligned}
\langle r\rangle \langle c_\alpha t - r\rangle^{1/2} |P_\alpha \partial \Gamma^a u(t,x)| &\le C[E_\kappa^{1/2}(u(t)) + \mathcal{X}_\kappa(u(t))], \quad |a|+3\le\kappa \\
\langle r\rangle \langle c_\alpha t - r\rangle |P_\alpha \partial \nabla \Gamma^a u(t,x)| &\le C\mathcal{X}_\kappa(u(t)), \quad |a|+4\le\kappa
\end{aligned}$$

These estimates and smallness allow you to bootstrap the nonlinearity in the bound of the weighted L^2 norm which will allow you to bound the weighted norm by the energy directly.

Similarly, you can obtain almost global existence from energy estimates this way. Based on

$$I \leq C\langle t\rangle^{-1}\langle r\rangle \sum_{\alpha=1,2}\langle c_\alpha t - r\rangle P_\alpha,$$

you can do $L^2 - L^\infty$ estimates with the weights given above.

To improve the almost global existence to global existence requires the null condition. You need an estimate of the form:

$$\begin{aligned}
|\langle u(x), N(v(x), w(x))\rangle| \leq \frac{C}{r}|u(x)| \sum_{|a|\leq 1} [|\nabla\tilde{\Omega}^a v(x)||\nabla w(x)| + |\nabla\tilde{\Omega}^a w(x)||\nabla v(x)| \\
+ |\nabla^2 v(x)||\tilde{\Omega}^a w(x)| + |\nabla^2 w(x)||\tilde{\Omega}^a v(x)|] \\
+ C\sum_{\mathcal{N}} |P_\alpha u(x)| \left[|P_\beta \nabla^2 v(x)||P_\gamma \nabla w(x)| + |P_\beta \nabla^2 w(x)||P_\gamma \nabla v(x)| \right],
\end{aligned}$$

where $\mathcal{N} = \{(\alpha, \beta, \gamma) \neq (1,1,1), (2,2,2)\}$ is the set of nonresonant indices.

If you are nonresonant than you can improve the bound, to for example something like

$$I \leq C \langle t \rangle^{-3/2} \langle r \rangle \langle c_1 t - r \rangle \langle c_2 t - r \rangle^{1/2},$$

You assume the smallness condition for one level of energy (the lower energy) and you use the null condition at this level, for the higher energy it is sufficient to have $\mathcal{O}(t^{-1})$ decay.

There are many difficulties when moving to an exterior domain. Since the exterior domain problem has been well studied for the wave equation of course many of the ideas that follow come directly from those previous works. Often the corresponding result for the wave equation can be proven for elasticity by a clever decomposition of the elasticity equations - whether by looking at the fundamental solution of the equations (like John) or by using a Helmholtz-Hodge decomposition.

Exponential decay of local energy:

Assume we have a nontrapping obstacle, $\mathcal{K} \subset \mathbb{R}^3$ with smooth boundary. Assume $0 \in \mathcal{K} \subset \{|x| < 1\}$, and u satisfies Dirichlet boundary conditions. Following Yamamoto '89 (based on the classical result for wave equations by Morawetz, Ralston, Strauss '77) we can obtain exponential decay of local energy.

If u solves $Lu = 0$, $u|_{\partial\mathcal{K}} = 0$, and $\operatorname{supp} u(0,\cdot), \partial_t u(0,\cdot) \subset \{|x| < 10\}$ then

$$\Big(\int_{\{x\in\mathbb{R}^3\setminus\mathcal{K}:|x|<10\}} |u'(t,x)|^2 dx\Big)^{1/2} \lesssim e^{-ct}\|u'(0,\cdot)\|_2.$$

We will need higher order versions of this which can be established using elliptic regularity. This was done for elasticity in Metcalfe '06 by combining methods from the wave equation with the Helmholtz-Hodge decomposition. Using elliptic regularity and the fact that ∂_t preserves the Dirichlet boundary conditions you can obtain the following:

Type of higher order local energy decay estimate needed:

If $u|_{\partial\mathcal{K}} = 0$, and $Lu(t,x) = 0$ for $|x| > 4$, and $t > 0$, if $u(t,x) = 0$ for $t \le 0$, then for M, ν fixed and c as in the linear case above,

$$\sum_{\substack{|\alpha|+\mu \le M+\nu \\ \mu \le \nu}} \|S^\mu \partial^\alpha u'(t,\cdot)\|_{L^2(|x|<4)} \lesssim \sum_{\substack{|\alpha|+\mu \le M+\nu-1 \\ \mu \le \nu}} \|S^\mu \partial^\alpha Lu(t,\cdot)\|_2 + \int_0^t e^{-(c/2)(t-s)} \sum_{\substack{|\alpha|+\mu \le M+\nu \\ \mu \le \nu}} \|S^\mu \partial^\alpha Lu(s,\cdot)\|_2 ds.$$

Energy Estimates:

The energy estimates are standard, but we need to keep track of the scaling vector field. This is due to the fact that coefficients of S can be arbitrarily large in a neighborhood of $\partial\mathcal{K}$, and so we have to restrict ourselves to fewer occurrences of S. This has been done in previous work of Keel-Smith-Sogge, Metcalfe-Sogge, etc.

For standard energy (no vector fields) elliptic regularity will suffice. At the next level of difficulty consider

$$\tilde{S} = t\partial_t + \eta(x) r\partial_r - 1$$

a modified scaling vector field where $\eta(x) \equiv 0$ for $x \in \mathcal{K}$, and $\eta(x) \equiv 1$ for $|x| > 1$. Then establish energy bounds for energies involving $\tilde{S}$. Estimate the commutator with the cut-off. Finally establish energy estimates for

$$Y_{N,\nu}(t) = \sum_{\substack{|\alpha|+\nu \le N+\nu \\ \mu \le \nu}} E_1(S^\mu Z^\alpha u(t,\cdot))$$

obtain roughly for solutions of $Lu = F$, if u vanishes for large $|x|$ and each t,

$$\partial_t Y_{N,\nu} \lesssim Y_{N,\nu}^{1/2} \sum_{\substack{|\alpha|+\mu\leq N+\nu \\ \mu\leq\nu}} \|LS^\mu Z^\alpha u(t,\cdot)\|_2 + \sum_{\substack{|\alpha|+\mu\leq N+\nu+1 \\ \mu\leq\nu}} \|S^\mu \partial^\alpha u'(t,\cdot)\|^2_{L^2(|x|<1)}.$$

Boundary terms which arise from standard energy method will look like

$$\sum_{\substack{|\alpha|+\mu\leq N_0+\nu_0 \\ \mu\leq\nu_0}} \int_{\partial\mathcal{K}} |e_k(S^\mu Z^\alpha u)(t,y)n_k|\, d\sigma(y)$$

with e_k the energy momentum vector for L, and this can be bounded by the RHS above using the fact that the obstacle $\mathcal{K} \subset \{|x| < 1\}$, and a trace theorem.

Pointwise Estimates:

Start with boundaryless equation: using spherical means express the solution in terms of the initial data; idea goes back to John '88. Obtain pointwise estimates like those for the wave equation using finite propagation speed. Let v be a solution to the linear homogeneous operator $Lv = 0$, Using a representation of v in terms of the initial data

$$(1+t+|x|)|v(t,x)| \lesssim \sum_{|\alpha|\leq 4} \|\langle x\rangle^{|\alpha|}\partial^\alpha f\|_2 + \sum_{|\alpha|\leq 3} \|\langle x\rangle^{1+|\alpha|}\partial^\alpha g\|_2.$$

Let w be solution to the inhomogeneous $Lw = G$, if $w = 0$ for $t \leq 0$ then, using Duhamel and Huygens' principle for L :

$$(1+t+|x|)|w(t,x)| \lesssim \sum_{\substack{|\alpha|+\mu\leq 3\\ \mu\leq 1}} \int_0^t \int_{\mathbb{R}^3} |S^\mu Z^\alpha G(s,y)| \frac{dy\, ds}{|y|}.$$

If $G \equiv 0$ for $|y| > 20c_1 s$, then

$$(1+t)|w(t,x)| \lesssim \sum_{\substack{|\alpha|+\mu\leq 3\\ \mu\leq 1}} \int_{\theta t}^t \int_{\mathbb{R}^3} |S^\mu Z^\alpha G(s,y)| \frac{dy\, ds}{|y|}, \quad \text{for } |x| < c_2 t/10,$$

where θ is some constant depending on c_2.

Pointwise Estimates in $\mathbb{R}^3\backslash\mathcal{K}$:

$$\begin{cases} Lu(t,x) = F(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3\backslash\mathcal{K} \\ u(t,x)|_{\partial\mathcal{K}} = 0. \\ u(0,\cdot) = f, \quad \partial_t v(0,\cdot) = g \\ \text{supp } f,g \in \{|x| \geq 6\} \end{cases}$$

Goal is to establish: Let $\mathcal{K} \subset \mathbb{R}^3$ be a bounded, nontrapping obstacle with smooth boundary. Then smooth solutions of the above satisfy

$$(1+t+|x|)|S^\nu Z^\alpha u(t,x)| \lesssim \sum_{\substack{j+|\beta|+k \leq M+\nu+7 \\ j \leq 1}} \|\langle x\rangle^{j+|\beta|} \nabla^\beta \partial_t^{k+j} u(0,\cdot)\|_2$$

$$+ \int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu \leq M+\nu+6 \\ \mu \leq \nu+1}} |S^\mu Z^\beta F(s,y)| \frac{dy\,ds}{|y|}$$

$$+ \int_0^t \sum_{\substack{|\beta|+\mu \leq M+\nu+3 \\ \mu \leq \nu+1}} \|S^\mu \partial^\beta F(s,\cdot)\|_{L^2(\{|x|<4\})}\, ds$$

Proof follows closely work of KSS '04 and Metcalfe-Sogge '05. Portions of the proof are also seen in Xin-Qin '04.

The first step is instead to bound instead:

$$(1+t+|x|)|S^{\nu}Z^{\alpha}u(t,x)| \lesssim \sum_{\substack{j+|\beta|+k\le M+\nu+4\\ j\le 1}} \|\langle x\rangle^{j+|\beta|}\nabla^{\beta}\partial_t^{k+j}u(0,\cdot)\|_2$$

$$+\int_0^t\int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{\substack{|\beta|+\mu\le|\alpha|+\nu+3\\ \mu\le\nu+1}} |S^{\mu}Z^{\beta}F(s,y)|\,\frac{dy\,ds}{|y|}$$

$$+\sup_{|y|\le 2,0\le s\le t}(1+s)\sum_{\substack{|\beta|+\mu\le|\alpha|+\nu\\ \mu\le\nu}}\Big(|S^{\mu}Z^{\beta}u'(s,y)|+|S^{\mu}Z^{\beta}u(s,y)|\Big).$$

This is clear for $|x| < 2$, so you can reduce this via a cut-off function ρ satisfying $\rho(r) \equiv 1$ for $r \geq 2$, and $\rho(r) \equiv 0$ for $r \leq 1$, we can establish the bound instead for $w(t,x) = \rho(|x|)S^\nu Z^\alpha u(t,x)$, which solves the boundaryless equation:

$$\begin{aligned} Lw = \rho L S^\nu Z^\alpha u - 2c_2^2 \nabla\rho \cdot \nabla(S^\nu Z^\alpha u) - c_2^2(\Delta\rho) S^\nu Z^\alpha u \\ - (c_1^2 - c_2^2)\nabla(\nabla\rho \cdot S^\nu Z^\alpha u) - (c_1^2 - c_2^2)\nabla\rho\nabla \cdot (S^\nu Z^\alpha u) \end{aligned}$$

and the estimates follow from the estimates for the boundaryless equation.

Finally we have to bound

$$(1+t) \sum_{\substack{|\beta|+\mu \le M+\nu+1 \\ \mu \le \nu}} \sup_{|x|<2} |S^{\mu}\partial^{\beta} u(t,x)|$$

by the RHS above. Using smooth cut-offs we can consider

Case 1: Vanishing data with an inhomogeneity $F(s,y)$ vanishing for $|y| > 4$, or

Case 2: $F(s,y) = 0$ for $|y| < 3$.

The first case will follow from the decay of local energy after using the Fundamental Theorem of Calculus and Sobolev embedding

$$
\begin{aligned}
(1+t)\sum_{\substack{|\beta|+\mu\le M+\nu+1\\ \mu\le\nu}} \sup_{|x|<2} |S^\mu \partial^\beta u(t,x)| \\
&\lesssim \int_0^t \sum_{\substack{|\beta|+\mu\le M+\nu+2\\ \mu\le\nu,\, j\le 1}} \|(s\partial_s)^j S^\mu \partial^\beta u'(s,\cdot)\|_{L^2(\{|x|<4\})}\, ds \\
&\lesssim \int_0^t \sum_{\substack{|\beta|+\mu\le M+\nu+3\\ \mu\le\nu+1}} \|S^\mu \partial^\beta u'(s,\cdot)\|_{L^2(\{|x|<4\})}\, ds.
\end{aligned}
$$

For case 2 one must further split the equation and it is here where the boundaryless estimates come in.

KSS Estimates:

These type of exterior domain estimates were first done for elasticity by Metcalfe '06, following work on boundaryless wave equation by Rodnianski '05 and adapted to a boundary by Metcalfe-Sogge. To adapt the proof from the wave equation to elasticity one can use a Helmoltz-Hodge decomposition. In order to take care of the boundary use the decay of local energy near the boundary and the boundaryless estimate for large $|x|$.

Suppose that $\mathcal{K} \subset \{|x| < 1\} \subset \mathbb{R}^3$ is a nontrapping obstacle with smooth boundary. Suppose further that $u \in C^\infty$ satisfies $u|_{\partial\mathcal{K}} = 0$,

$u(t,x)=0$ for $t\le 0$, and vanishes for large $|x|$ for every t. Then,

$$(\log(2+T))^{-1/2}\sum_{\substack{|\alpha|+\mu\le M+\nu\\ \mu\le\nu}}\|\langle x\rangle^{-1/2}S^{\mu}Z^{\alpha}u'\|_{L^2_tL^2_x(S_T)}$$

$$\lesssim\int_0^T\sum_{\substack{|\alpha|+\mu\le M+\nu\\ \mu\le\nu}}\|S^{\mu}Z^{\alpha}Lu(s,\cdot)\|_2\,ds+\sum_{\substack{|\alpha|+\mu\le M+\nu-1\\ \mu\le\nu}}\|S^{\mu}Z^{\alpha}Lu\|_{L^2_tL^2_x(S_T)}$$

for any fixed M,ν and $T>0$. Here S_T is the time strip $[0,T]\times\mathbb{R}^3\backslash\mathcal{K}$.

Weighted Sobolev and Null form estimates:

Now we need the weighted L^2 estimates we saw before in the presence of a boundary.

$$\sum_{a=1,2} \| \langle c_a t - r\rangle P_a \partial \nabla S^\nu Z^\alpha u(t,\cdot)\|_2 \lesssim \sum_{\substack{|\beta|+\mu\le M+\nu+1\\ \mu\le \nu+1}} \|S^\mu Z^\alpha u'(t,\cdot)\|_2 + t \sum_{\substack{|\beta|+\mu\le M+\nu\\ \mu\le\nu}} \|S^\mu Z^\beta L u(t,\cdot)\|_2 + (1+t)\sum_{\mu\le\nu} \|S^\mu u'(t,\cdot)\|_{L^2(|x|<2)}$$

Coefficients of Z are in control on $\{|x| < 3/2\}$, so apply a cut-off function and the boundaryless estimates and be careful with the commutator. This will give all the weighted L^2 and L^∞ estimates as in the whole space case, with the corresponding extra terms from the boundary.

The Null form estimates are exactly the same as in the full space case.

Completion of proof:

In the end one must do a reduction to avoid difficulties with the compatibility conditions and to deal with an equation which meets the requirements of the auxiliary lemmas we have developed. We fix a smooth cutoff function η with $\eta(t,x) \equiv 1$ if $t \leq 3/2$ and $|x| \leq 20$, $\eta(t,\cdot) \equiv 0$ for $t > 2$, and $\eta(\cdot,x) \equiv 0$ for $|x| > 25$. Setting $u_0 = \eta u$, which by local existence theory is known and satisfies a local bound, it follows that u solves the original problem for $0 < t < T$ if and only if $w = u - u_0$ solves

$$\begin{cases} Lw = (1-\eta)Q(\nabla u, \nabla^2 u) - [L,\eta]u, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}, \\ w|_{\partial\mathcal{K}} = 0, \\ w(0,\cdot) = (1-\eta)(0,\cdot)f, \\ \partial_t w(0,\cdot) = (1-\eta)(0,\cdot)g - \eta_t(0,\cdot)f \end{cases}$$

over the same interval.

We now fix another smooth cutoff β with $\beta(z) \equiv 1$ for $z \geq 10$ and $\beta(z) \equiv 0$ for $z \leq 6$. Then, let v solve the linear equation

$$\begin{cases} Lv = \beta\big(\frac{|x|}{t+1}\big)(1-\eta)Q(\nabla u, \nabla^2 u) - [L,\eta]u, \\ v|_{\partial\mathcal{K}} = 0, \\ v(0,\cdot) = (1-\eta)(0,\cdot)f, \\ \partial_t v(0,\cdot) = (1-\eta)(0,\cdot)g - \eta_t(0,\cdot)f. \end{cases}$$

We can get nice estimates for w, and v and for the most part restrict our attention to $w-v$ which satisfies

$$\begin{cases} L(w-v) = (1-\beta)\Big(\frac{|x|}{t+1}\Big)(1-\eta)(t,x)Q(\nabla u, \nabla^2 u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}, \\ (w-v)|_{\partial\mathcal{K}} = 0, \\ (w-v)(t,x) = 0, \quad t \leq 0. \end{cases}$$

This equation meets the requirements of our previous estimates. In particular, it has vanishing Cauchy data and a forcing term which is supported in $|x| \leq 20c_1 t$. Depending on the estimates being utilized, we shall use certain L^2 and L^∞ bounds for u while at other times we shall use them for $w-v$ or w. Since $u=(w-v)+v+u_0$ and since u_0 and v satisfy the proper smallness conditions, it will always be the case that a bound for $w-v$, w, or u will imply the same bound for the others.

The smallness conditions are:

(1) if ε in the theorem is sufficiently small, then there is a constant C_0 for which

$$\sup_{0\le t\le 2}\sum_{|\alpha|\le 112}\|\partial^\alpha u(t,\cdot)\|_{L^2(|x|\le 25)}\le C_0\varepsilon.$$

This follows from well-known local existence theory.

(2) On the other hand, over $\{|x|>5(t+1)\}$, u corresponds to a boundaryless solution. Thus by the estimates which correspond

to those that follow for the boundaryless problem, we have

$$\sup_{0\le t<\infty} \sum_{|\alpha|+\mu\le 111} \|S^\mu Z^\alpha u'(t,\cdot)\|_{L^2(|x|\ge 5(t+1))} + \sup_{\substack{|x|\ge 5(t+1)\\ 0\le t<\infty}} (1+t+|x|) \sum_{|\alpha|+\mu\le 108} |S^\mu Z^\alpha u'(t,x)| \le C_1\varepsilon.$$

(3) Finally, if v is the solution to the linear equation above, then

$$(1+t+|x|) \sum_{|\alpha|+\mu\le 102} |S^\mu Z^\alpha v(t,x)| + \sum_{|\alpha|+\mu\le 100} \|S^\mu Z^\alpha v'(t,\cdot)\|_2 + (\log(2+t))^{-1} \sum_{|\alpha|+\mu\le 98} \|\langle x\rangle^{-1/2} S^\mu Z^\alpha v'\|_{L^2_t L^2_x(S_t)} \le C_2\varepsilon$$

for some absolute constant $C_2 > 0$ and for any $t > 0$. This follows from the pointwise estimates and the weighted Sobolev estimates.

Completion of proof follows standard continuity argument, ours closely follows Metcalfe-Nakamura-Sogge ’05 which gives global existence for multiple speed wave equations in an exterior domain.