Pointwise Carleman estimates and control theoretic implications

(Joint work with Professor Roberto Triggiani)

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Outline of Talk:

1. Qualitative statement of exact controllability of an evolution equation

2. Background and some history of E.C. of Schrödinger-type equations

3. Problem setting, geometric assumptions on triple $\{M, \Gamma_0, \Gamma_1\}$.

4. Pointwise Carleman Inequality.

5. Carleman estimates: first version: Global uniqueness.

6. Carleman estimates: second version: Continuous observability.

7. The Euler-Bernoulli Plate Equation.

Qualitative statement of exact controllability of an evolution equation.

Evolution $y(t,x;u)$: a solution of differential equations on $[0,T]\times M$.

I. T arbitrary, infinite speed of propagation.

(Schrödinger equation, Euler-Bernoulli Plate Equation, $\cdots$)

II. An initial condition y_0 at $t=0$ and a target state y_T at $t=T$.

III. u a control function: acting on a portion of the boundary.

Exact controllability problem: Seek the control u such that:

(Initial condition y_0)$\rightarrow \left[\text{ evolution } y(t,\cdot;u)\right] \rightarrow$ (target state y_T)

Example of **NO** exact control from boundary: the evolution equation on a manifold with a closed geodesic.

For Schrödinger equations with Dirichlet control on a bounded domain Ω:

$$\begin{cases} iy_t + \mathcal{A}y = 0 & on\ \Omega \\ y|_{\Gamma_0} = 0,\ y|_{\Gamma_1} = u \in L^2([0,T] \times \Gamma_1), & \partial\Omega = \Gamma_0 \cup \Gamma_1 \end{cases}$$

Exact controllability at T on right space (H^{-1})

$\Updownarrow$

Solution operator being **ONTO** target space (H^{-1})

$\Updownarrow$

Dual operator being bounded below

$\Updownarrow$

For the dual problem: Schrödinger equation with homogeneous Dirichlet B.C.:

$$\begin{cases} iw_t + \mathcal{A}^*w = 0 & on\ \Omega \\ w|_{\partial\Omega} = 0 \end{cases}$$

PDE interpretation (**Continuous Observability Inequality**):

$$C_T E(0) \leq \int_0^T \int_{\Gamma_1} \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma_1 dt \qquad \textbf{(GOAL)}$$

(This is an **inverse type inequality**.)

Classic energy method for pure Schrödinger equations

Pure Schrödinger equation: $iy_t - \Delta y = 0$:

• Dirichlet boundary control +optimal regularity. [Lasiecka-Triggiani '91]

• Neumann boundary control. [Machtyngier '90]

• Optimal regularity and exact controllability of wave, Schrödinger, plate-like eq... [Ho, Lasiecka, Triggiani, Lions, Lagnese, etc].

Classic energy method failed.....

• Δ replaced by a variable coefficient elliptic operator, $\mathcal{A} = \sum \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$;

• Constant coefficient principal part with the presence of "energy level" terms:

$$iy_t - \Delta y = r(t,x) \cdot \nabla y + b(t,x)y.$$

Integral Carleman estimates with l.o.t.

• Geometric optics methods for pure Schrödinger equations with Dirichlet control. [Lebeau '91]

• Integral Carleman estimates:

• general evolution equations in pseudo-differential setting; [Tataru's Thesis at UVA '92 and series papers later];

• hyperbolic equations, Schrödinger, plates, etc, with differential energy methods [Lasiecka-Triggiani]

• Riemannian geometric versions of the above "concrete" energy methods, [Lasiecka-Triggiani-Yao]

An additional obstacle:

Integral Carleman estimates: polluted by interior l.o.t. below the energy level.

$$C_T E(0) \leq \int_0^T \int_{\Gamma_1} \left| \frac{\partial w}{\partial \nu} \right|^2 d\Gamma_1 dt + \mathbf{l.o.t.}(||w||^2_{L^2([0,T]\times M)})$$

Approach to absorbing l.o.t: Compactness/uniqueness method

- PDE theory (e.g. Hörmander's books)
- Control theory of PDE first by Littman,

Contradiction argument using a global uniqueness theorem for over-determinant problems.

- For analytic coefficients, one can use Holmgren-John semi-global result.
- For partial analytic coefficients, recent works by Tataru, Hörmander.

Could NOT do:

In the presence of **low regularity coefficients**, possibly also time-dependent, such uniqueness result was generally not available.

Pointwise Carleman estimates since late 90's

- Inspired by Novosibirsk school for global uniqueness for 2nd order hyperbolic equations on Euclidean domain with Dirichlet B.C. case. [Lavrentev-Romanov-Shishataskii '86]

- Global uniqueness, observability and stabilization for 2nd order hyperbolic equations on Euclidean domain with purely Neumann B.C. case and mixed B.C. case. [Lasiecka-Triggiani-Zhang '00]

- Global uniqueness, observability and stabilization for 2nd order hyperbolic equations on Riemannian manifold with purely Neumann B.C. case and mixed B.C. case. [Triggiani-Yao '02]

- Global uniqueness, observability and stabilization for Schrödinger equations on Euclidean domain with purely Neumann B.C. case and mixed B.C. case. [Lasiecka-Triggiani-Zhang '04]

Problem setting

(M,g) Riemannian manifold; $\partial M = \Gamma = \overline{\Gamma_0 \cup \Gamma_1}$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. Given $T > 0$,

$$\mathcal{P}w = iw_t + \Delta_g w = F(w, \bar{w}, \nabla w, \nabla \bar{w}) + f, \ in\ Q = (0,T] \times M \tag{1}$$

Linear: $F = (P(t,x), \nabla w) + p_0(t,x)w$ with $|P|$, $p_0 \in L_\infty(Q)$, and $f \in L_2(Q)$.

Semilinear: $|F(w, \bar{w}, \nabla w, \nabla \bar{w})|^2 \leq C(|\nabla w|^2 + |w|^{2p})$, with $p < n/(n-2)$, $n \geq 3$, and $p < \infty$, $n = 1, 2$.

For $\Sigma = (0,T] \times \Gamma$, $\Sigma_1 = (0,T] \times \Gamma_1$, consider the boundary conditions:

(i) Neumann B.C.: $\frac{\partial w}{\partial \nu}|_\Sigma = 0$, and $w|_{\Sigma_1} = u$,

(ii) Dirichlet B.C.: $w|_\Sigma = 0$, and $\frac{\partial w}{\partial \nu}|_{\Sigma_1} = u$.

Observation u in present problem (or **control** in the dual problem) takes place only on a **sub-portion** Γ_1 of the boundary.

Geometrical assumption on triple $\{M, \Gamma_0, \Gamma_1\}$

$\exists$ a strictly **convex** (w.r.t. Riemannian metric) C^3 function $d : \overline{M} \to \mathbf{R}^+$, s.t. for the conservative gradient field $h(x) = \nabla d(x)$:

(I) Neumann B.C.: $\frac{\partial d}{\partial \nu} = \nabla d \cdot \nu = 0$, on Γ_0.

Dirichlet B.C.: $\frac{\partial d}{\partial \nu} = \nabla d \cdot \nu \leq 0$, on Γ_0.

(II) Hessian of $d(x)$ is **coercive**:

$D^2 d(X, X) = (D_X(\nabla d), X) \geq 2(X, X), \forall x \in M, X \in T_x M$

A **temporary assumption**: no critical point of $d(x)$ on M (enough, near Γ_0),

$$\inf_M |\nabla d| = p > 0$$

To remove above assumption, splitting M as $M = M_1 \cup M_2$, for two suitably overlapping sub-manifolds M_1 and M_2, and working with two strictly convex functions d_1 and d_2, where d_i satisfies above assumption on M_i.

Examples

1. A bounded domain $\Omega \subset \mathbf{R}^n$ satisfies

(i) convex (respectively, concave) on the side of the portion Γ_0 of its boundary,

(ii) there exists a radial vector field $(x - x_0)$ for some $x_0 \in \mathbf{R}^n$ which is entering (respectively, exiting) Ω through Γ_0.

Convex function: $d(x) = \frac{1}{2}||x - x_0||^2 + \cdots$

2. Riemannian manifolds:

• Portion of totally geodesic ball with partial geodesic flat boundary on non-negative curvature manifolds;

• Submanifold with partial geodesic flat boundary on negative curvature manifolds.

Pseudo-convex function $\phi(t,x)$ on Q

Define function $\phi(t,x)$ on $Q=[0,T)\times M$ as

$\phi(t,x)=d(x)-c(t-\frac{T}{2})^2,\ 0\le t\le T,\ x\in M$

where $c=c_T$ large enough s.t. $cT^2>4\max_M d(x)+4\delta$

for sufficiently small $\delta>0$, and fixed. $\phi(t,x)$ satisfies:

(I) $\phi(0,x)=\phi(T,x)=d(x)-\frac{cT^2}{4}\le-\delta,\ uniformly\ for\ x\in M$

(II) there are $0<t_0<\frac{T}{2}<t_1<T$, such that $\min_{x\in M,t\in[t_0,t_1]}\phi(t,x)>-\frac{\delta}{2}$

$$E(t)=\int_M|\nabla w(t)|^2dx;$$
$$\mathcal{E}(t)=\int_M[|\nabla w(t)|^2+|w(t)|^2]dx=||w(t)||^2_{H^1(M)}.$$

Fundamental technical Lemma

Assume $w(t,x) \in C^2(Q,\mathbf{C})$, $l(t,x) \in C^3(Q,\mathbf{R})$, $\Phi(t,x)$, $\Psi(t,x) \in C^1(Q,\mathbf{R})$, with $\nabla_x(l_t) \equiv 0$; $\theta(t,x) = e^{l(t,x)}$; $v(t,x) = \theta(t,x)w(t,x)$.

Let $\epsilon > 0$ arbitrary, the following pointwise inequality holds true:

$$\begin{aligned}
& (1+\frac{1}{\epsilon})e^{2l(t,x)}|iw_t + \Delta w|^2 - \frac{dM}{dt} + divV \\
\geq \ & \Big\{ -2(\Psi + \Delta l)\Delta l + 4D^2 l(\nabla l, \nabla l) + 2(\nabla(\Phi - \Psi), \nabla l) - \epsilon|\Psi + \Delta l|^2 \\
& -\frac{1}{\epsilon}|\nabla(\Delta l)|^2 - 4(\nabla l, \nabla(\Delta l)) - \Psi^2 - \Phi^2 + 2\Phi\Delta l + l_{tt} \Big\}|v|^2 \\
& +2\Big\{ D^2 l(\nabla v, \nabla \bar{v}) + D^2 l(\nabla \bar{v}, \nabla v) - (\Psi + \Delta l)|\nabla v|^2 \Big\} - \epsilon|\nabla v|^2 \qquad (2)
\end{aligned}$$

where $M(w)$ and $V(w)$ have explicit formula. Let $\xi \equiv Rew$ and $\eta \equiv Imw$:

$$M(w) \equiv \theta[2(\nabla l, \nabla \xi)\eta - \xi(\nabla l, \nabla \eta) - l_t|w|^2];$$

$$\begin{aligned}
V(w) \ \equiv \ & \theta^2\{(2|\nabla l|^2 - \Delta l - \Psi + \Phi)\nabla l|w|^2 + l_t(\eta\nabla\xi - \xi\nabla\eta) - \nabla l(\xi_t\eta - \xi\eta_t) \\
& +\frac{1}{2}(2|\nabla l|^2 - \Delta l)\nabla|w|^2 + (\nabla l, \nabla\bar{w})\nabla w + (\nabla l, \nabla w)\nabla\bar{w} - \nabla l|\nabla w|^2\}.
\end{aligned}$$

Pointwise Carleman Inequality

Choose $l(t,x) = \tau\phi(t,x)$, $\Phi(t,x) = -\Delta l(t,x)$, either $\Psi(t,x) = \Delta l(t,x)$ or $\Psi(t,x) = 0$ in above Lemma, we have

$$
\begin{aligned}
& \big(1+\frac{1}{\epsilon}\big)e^{2\tau\phi(t,x)}|iw_t+\Delta w|^2 - \frac{dM}{dt} + divV \\
\geq \quad & \left\{2\tau\left[D^2d(\frac{\nabla v}{|\nabla v|},\frac{\nabla\bar{v}}{|\nabla v|}) + D^2d(\frac{\nabla\bar{v}}{|\nabla v|},\frac{\nabla v}{|\nabla v|})\right] - \epsilon\right\}|\nabla v|^2 \\
& + \left[4\tau^3 D^2 d(\nabla d,\nabla d) + O(\tau^2)\right]|v|^2 \\
\geq \quad & \{4\tau\rho - \epsilon\}|\nabla v|^2 + [4\tau^3 p^2 + O(\tau^2)]|v|^2 \\
\geq \quad & \delta_0\{4\tau\rho - \epsilon\}\theta^2|\nabla \mathbf{w}|^2 + [4\tau^3 p^2(1-\delta_0) + O(\tau^2)]\theta^2|\mathbf{w}|^2 \qquad (3)
\end{aligned}
$$

for some $0 < \delta_0 < 1$. Note that:

- $D^2d(\cdot,\cdot)$ positive definite $\Rightarrow$ for small $\epsilon > 0$, the coefficient of $|\nabla v|^2$ positive;
- $\tau > 0$ large enough and $d(x)$ no critical point $\Rightarrow$ the coefficient of v^2 positive.

Carleman estimates: first version

Integrate (3) over $Q = [0,T] \times M$, applying the assumption on $F(w)$, we have the following **first version Carleman estimates**:

$$
\begin{aligned}
& B_\Sigma(w) + (1+\frac{1}{\epsilon}) \int_0^T \int_M e^{2\tau\phi(t,x)} [|\mathbf{F}|^{\mathbf{2}} + |f|^2] dx dt \\
\geq \quad & m_\tau \int_0^T \int_M e^{2\tau\phi(t,x)} [|\nabla w|^2 + |w|^2] dx dt - c\tau e^{-2\delta\tau} [\mathcal{E}(T) + \mathcal{E}(0)] \\
\geq \quad & m_\tau e^{-\delta\tau} \int_{t_0}^{t_1} \mathcal{E}(t) dt - C\tau e^{-2\delta\tau} [\mathcal{E}(T) + \mathcal{E}(0)]. \qquad (4)
\end{aligned}
$$

where $m_\tau \to \infty$ as $\tau \to \infty$ at the growth rare as τ, the boundary term

$$
B_\Sigma(w) \quad = \quad \int_0^T \int_M div V dx dt = \int_0^T \int_\Gamma V \cdot \nu d\Gamma dt
$$

$$
\int_Q \frac{\partial M}{\partial t} dt dx = \left[\int_M M dx \right]_0^T \leq \tau C \left[\int_M e^{2\tau\phi} (|\nabla w|^2 + |w|^2) dx \right]_0^T \leq C\tau e^{-2\tau\delta} [\mathcal{E}(T) + \mathcal{E}(0)].
$$

Global uniqueness

Theorem (Global uniqueness) [Triggiani-Xu '07]

Let $w \in H^{2,2}(Q) = L_2(0,T;H^2(M)) \cap H^2(0,T;L_2(M))$ be a solution of (1) and $f = 0$, with $\Sigma = [0,T] \times \Gamma$, $\Sigma_1 = [0,T] \times \Gamma_1$,

(I) Neumann case:w satisfies the B.C.:

$\frac{\partial w}{\partial \nu}|_{\Sigma} = 0, \; and \; w|_{\Sigma_1} = 0, \; where \; h \cdot \nu = 0 \; on \; \Gamma_0.$

then such a solution must vanish: $w = 0$ in $[0,T) \times M$.

(II) Dirichlet case:w satisfies the B.C.:

$w|_{\Sigma} = 0, \; and \; \frac{\partial w}{\partial \nu}|_{\Sigma_1} = 0, \; where \; h \cdot \nu \leq 0 \; on \; \Gamma_0.$

then such a solution must vanish: $w = 0$ in $[0,T) \times M$.

Sketch of proof: Carleman estimates $\Rightarrow$ Global uniqueness

- Step 1. From definition of $B_\Sigma(w)$, one has

$$\begin{cases} Neumann\ B.C. \Rightarrow B_\Sigma(w) = 0 \\ Dirichlet\ B.C. \Rightarrow B_\Sigma(w) = 2\tau \int_0^T \int_{\Gamma_0} e^{2\tau\phi} |\frac{\partial w}{\partial \nu}|^2 h \cdot \nu \leq 0. \end{cases}$$

- Step 2. With $B_\Sigma(w) \leq 0$ and $f = 0$, from Carleman estimate (4), one has

$$0 \geq m_\tau e^{-\delta\tau} \int_{t_0}^{t_1} \mathcal{E}(t)dt - C\tau e^{-2\delta\tau}[\mathcal{E}(T) + \mathcal{E}(0)]$$

$$i.e. \qquad \int_{t_0}^{t_1} \mathcal{E}(t)dt \leq \frac{C\tau e^{-2\delta\tau}[\mathcal{E}(T) + \mathcal{E}(0)]}{m_\tau e^{-\delta\tau}}$$

Let $\tau \nearrow \infty$, one has $m(\tau) \nearrow \infty$ at the rate of τ, we conclude that

$\int_{t_0}^{t_1} \mathcal{E}(t)dt = 0 \Rightarrow w = 0$ on $(t_0, t_1) \times M$.

- Step 3. Extend $(t_0, t_1) \rightarrow [0, T]$: replace (t_0, t_1) by a large time interval where $\phi(t, x) \geq \sigma > -\delta$ uniformly in M, with $\sigma \rightarrow -\delta$. $\Rightarrow w = 0$ on $(0, T) \times M$;

$w = 0$ on $[0, T] \times M$ from $w \in C([0, T]; L^2(M))$, *a-fortiori* from $w \in H^{2,2}(Q)$.

Carleman estimates: second version

• Assume $P(t,x)$ is purely imaginary (as in the case of magnetic potential). (while dim = 1, no need to assume $P(t,x)$ purely imaginary)

• Energy method: multiply (1) by $i[\Delta\bar{w}-\bar{w}]$, take real parts, $\Rightarrow |E(t)-E(s)| \leq G(T) + c_T \int_s^t E(\sigma)d\sigma$. where $G(T) = C||f||^2_{L_2(0,T;H^1(M))} + boundary\ terms$.

• Apply Gronwall inequality $\Rightarrow E(t) \geq \frac{E(T)+E(0)}{2}e^{-c_T T} - G(T)$; $0 \leq t \leq T$.

First version Carleman estimates⇒**second version Carleman estimates**:

$$
\begin{aligned}
& \bar{B}_\Sigma(w) + (1+\frac{1}{\epsilon})\int_Q e^{2\tau\phi}|f|^2 dxdt + C||f||^2_{L_2(0,T;H^1(M))} \\
\geq\ & \left\{ m_\tau e^{-\delta\tau}\frac{t_1-t_0}{2}e^{-c_T T} - c\tau e^{-2\delta\tau}\right\}[E(T)+E(0)] \\
\geq\ & k_{\phi,\tau}[E(T)+E(0)]. \qquad (5)
\end{aligned}
$$

where the new boundary term:

$\bar{B}_\Sigma(w) = B_\Sigma(w) + boundary\ terms\ from\ Gronwall\ inequality.$

Extension of Carleman estimates to finite energy solutions

Approximation by smooth solutions to extend all previous estimates from $H^{2,2}(Q)$ solutions to finite energy solutions in the the class

$$w \in C(0,T; H^1(M)),\ w_t, \frac{\partial w}{\partial \nu} \in L_2(0,T; L_2(\Gamma)).$$

I. Purely Dirichlet case: $\frac{\partial w}{\partial \nu} \in L^2(\Gamma)$ for $w_0 \in H_0^1(M)$ from the optimal regularity result [Lasiecka-Triggiani '91]

II. Purely Neumann case:

• Difficult: no H^1-traces on boundary for finite energy solutions

• Strategy: regularization procedure as [First by Lasiecka-Tataru '93, Lasiecka-Triggiani-Zhang '00](2nd order hyperbolic equations) + an unbounded perturbation of the basic generator on the state space $H^1(M)$.

Continuous observability for Dirichlet B.C.

Consider the purely Dirichlet B.C. problem:

$$\begin{cases} iw_t + \Delta_g w = F(w) + f, \ in\ Q, \\ w(0,x) = w_0(x), \quad in\ M, \\ w|_\Sigma = 0, \quad in\ \Sigma = [0,T] \times \Gamma. \end{cases} \tag{6}$$

Theorem:[Triggiani-Xu '07] Let w be a solution of (6) with I.C. $w_0 \in H_0^1(M)$ and with $f \in L_2(0,T; H^1(M))$, under assumptions on $F(w)$ and geometrical conditions on $d(x)$ (here only need $\nabla d \cdot \nu \leq 0$, on Γ_0). Then there exists a constant $C_T > 0$, the **continuous observability inequality** is true:

$$C_T E(0) \leq \int_0^T \int_{\Gamma_1} |\frac{\partial w}{\partial \nu}|^2 d\Gamma_1 dt + ||f||^2_{L_2(0,T;H^1(M))}.$$

where C_T has explicit formula, useful for nonlinear problems. And C_T is of order Ce^{-CL^2}, where L is the appropriate norm of the coefficients $P(t,x)$ and $p_0(t,x)$, for $n \geq 3$, one has

$$L = |p_0|_{L^\infty(M)} + |p_0|_{L^1(0,T;W^{1,n}(M))} + |P|_{L^\infty(0,T;W^{1,\infty}(M^n))}. \tag{7}$$

and analogously for $n = 1, 2$.

Continuous observability for Neumann B.C.

Consider the purely Neumann B.C. problem:

$$\begin{cases} iw_t + \Delta_g w = F(w) + f, \ in \ Q, \\ w(0,x) = w_0(x), \quad in \ M, \\ \frac{\partial w}{\partial \nu}|_{\Sigma} = 0, \quad in \ \Sigma = [0,T] \times \Gamma. \end{cases} \tag{8}$$

Theorem:[Triggiani-Xu '07] Let w be a solution of (8) with I.C. $w_0 \in H^1(M)$ and with $f \in L_2(0,T;H^1(M))$, under assumptions on $F(w)$ and geometrical conditions on $d(x)$. Then there exists a constant $C_T > 0$, the **continuous observability inequality** is true:

$$C_T E(0) \leq \int_0^T \int_{\Gamma_1} [|w|^2 + |w_t|^2] d\Gamma_1 dt + ||f||^2_{L_2(0,T;H^1(M))}.$$

where C_T has explicit formula as the purely Dirichlet B.C. case (7).

The Euler-Bernoulli Equation

Consider the following Euler-Bernoulli plate problem:

$$\begin{cases} w_{tt} + \Delta_g^2 w = F(w, \nabla w, \Delta w) + f, \ in\ Q = (0,T] \times M, \\ w(0,x) = w_0(x), \ w_t(t,x) = w_1(x), \quad in\ M, \\ w|_\Sigma = 0, \ \Delta w|_\Sigma = 0 \quad in\ \Sigma = [0,T] \times \Gamma. \end{cases} \tag{9}$$

Writing the E-B equation as an iteration of two Schrödinger equations:

$$w_{tt} + \Delta^2 w = (\Delta + i\partial_t)(\Delta - i\partial_t)w.$$

Setting $v = iw_t - \Delta w$, rewrite problem (9) as

$$\begin{cases} iv_t + \Delta v = F + f, \ in\ Q = (0,T] \times M, \\ v(0,x) = iw_1(x) - \Delta w_0(x), \quad in\ M, \\ v|_\Sigma = 0, \quad in\ \Sigma = [0,T] \times \Gamma. \end{cases} \tag{10}$$

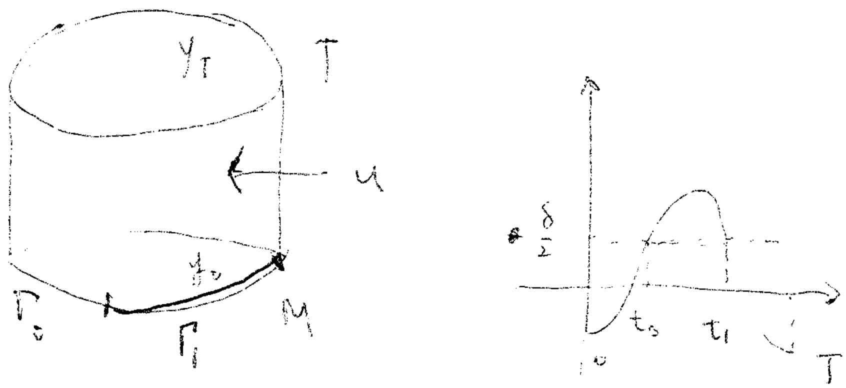
Setting $E_w(t) = \int_M [|\nabla \Delta w(t)|^2 + |\nabla w_t(t)|^2] dx = E_v(t) = \int_M |\nabla v(t)|^2 dx.$

Theorem:[Triggiani-Xu '07] Let w be the solution of problem (9) with $\{w_0, w_1\} \in H^3(M) \times H^1(M)$. Let $f \in L^2(0,T; H^1(M))$. Let $d(x)$ be the strictly convex function satisfying above geometric assumptions. Then there exists a constant $C_T > 0$, the **continuous observability inequality** holds true:

$$C_T E_w(0) \leq \int_0^T \int_{\Gamma_1} \left[\left(\frac{\partial \Delta w}{\partial \nu} \right)^2 + \left(\frac{\partial w_t}{\partial \nu} \right)^2 \right] d\Gamma_1 dt + ||f||^2_{L^2(0,T;H^1(M))}.$$

where C_T has explicit formula as the purely Dirichlet B.C. case (7).

Thanks!



$$|F|^2 \le C(|\nabla w|^2 + |w|^{2p})$$

$$\int_\Omega |w|^{2p} \le \int_\Omega |\nabla w|^2 + |w|^2$$

$$\int |F|^2 \le \int_\Omega (|\nabla w|^2 + |w|)$$

Bourgain '97 $\quad (i\partial_t + \Delta)u = F(u, \bar{u}, \nabla u, \nabla \bar{u}) \quad \mathbb{R}^n \times \mathbb{R}$

F poly $\Rightarrow$ has no non-zero compact support sol

Ionescu Kenig 06

if $u \equiv 0$ outside $\{|x| \le 0\} \times \{0, 1\}$

$\Rightarrow u = 0$