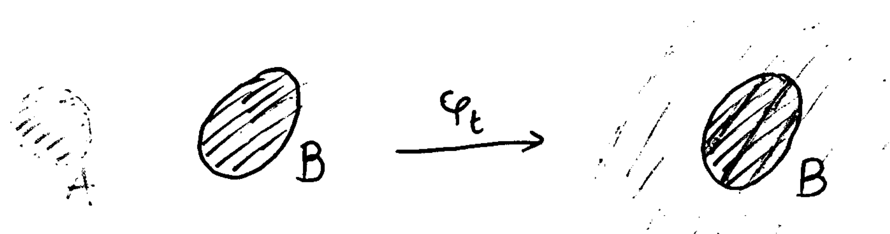
Introduction

- One of the *goals* in **dynamical systems**: understand the long time evolution of a system which displays a *chaotic* behavior;
- Setting of **ergodic theory**: the evolution is given by measure-preserving transformations:
  on a measure space $(X, \mathscr{A}, \mu)$, $\mu(X) < \infty$ consider:
  $(\mu(X) = 1)$
  - *discrete* dynamical system: iterations of a measure preserving automorphism $T : X \to X$;
    T is *measure preserving* if $\forall\ A \subset X$ measurable $(A \in \mathscr{A})$, $\mu(T^{-1}A) = \mu(A)$;
  - *continuous* dynamical system: group of measure preserving automorphisms $\{\varphi_t\}_{t\in\mathbb{R}}$, $\varphi_{t+s} = \varphi_t \cdot \varphi_s$;
- Examples of ergodic properties:
  - *ergodicity*: there are no non-trivial invariant sets, i.e., if $\varphi_t(A) = A\ \forall t \in \mathbb{R}$, then $\mu(A) = 0$ or 1;
  - *mixing*: asymptotic uniform distribution:

$$\boxed{\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B),} \qquad \forall A, B \in \mathscr{A}.$$



Motivation

Ergodic properties of a class of area-preserving flows on surfaces:

Flows given by multi-valued Hamiltonians (Novikov)

S compact surface of genus $g \geq 2$;

ω closed 1-form, $\omega \overset{loc}{=} \mathrm{dH}$

φ_t Hamiltonian flow given by H

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$$

Assume ω is *Morse*; φ_t is given by a vector field with centers and simple saddles;

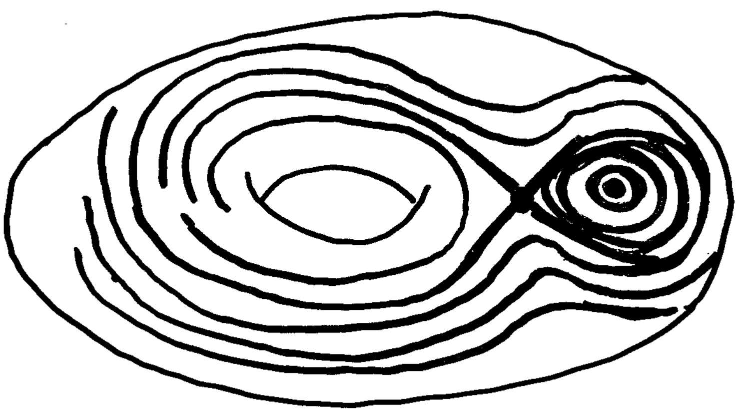
Decomposition: (Zorich, Levitt) (Maier)
- components filled by periodic orbits;
- minimal components (up to g)

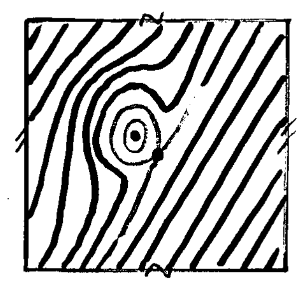
Mixing

Consider the torus $\mathbb{T}^2$, ω Morse closed 1-form;

- ***Question:*** (Arnold)

Is the flow on the minimal component mixing?
$(\mu(\varphi_t(A) \cap B) \xrightarrow{t\to\infty} \mu(A)\mu(B)\ \forall A, B$ measurable)

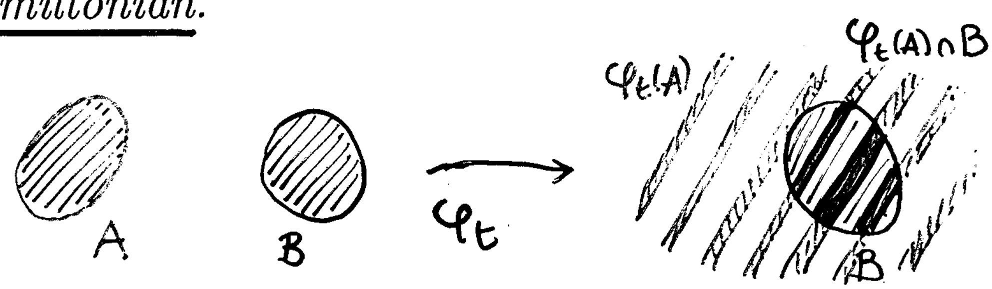




Yes. *(generically)* (Khanin-Sinai)

- ***Question:*** Are the minimal components of a flow on S of $g \geq 2$ mixing?

Remark. *It is essential to consider the parametrization given by the multi-valued Hamiltonian.*



Suspension Flows

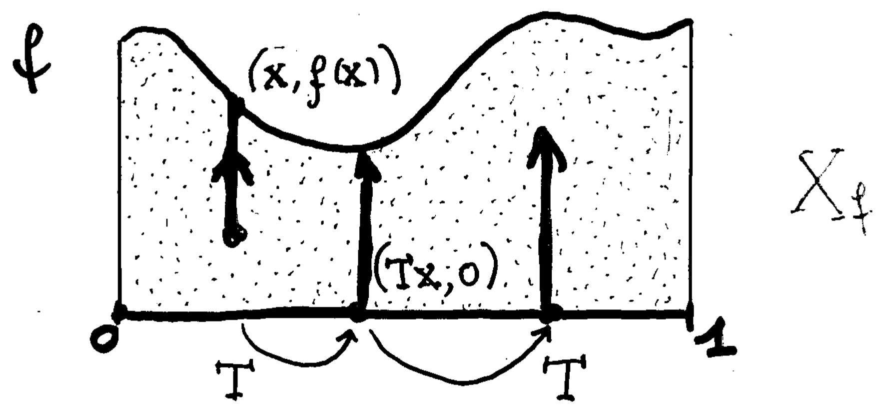
Start with:

a) a measure preserving $\boldsymbol{T} : [0,1] \to [0,1]$;

b) a roof function $\boldsymbol{f} : I \to \mathbb{R}^+$, $\int_0^1 f \mathrm{d}x = 1$;

Definition. *The suspension flow* $\{\boldsymbol{\varphi_t}\}_{t\in\mathbb{R}}$ *built over T under the roof f is defined on:*

$$\boldsymbol{X_f} \doteq \{(x,y) \mid \quad x \in I, \quad 0 \le y < f(x)\}$$



and its action is generated by:

$$\begin{cases} \varphi_t(x,y) & = & (x, y+t), \text{ if } 0 \le y+t < f(x); \\ \varphi_{f(x)}(x,0) & = & (Tx, 0). \end{cases}$$

Remark. *If T preserves $\mathrm{d}x$, φ_t preserves*

$$\boldsymbol{\mu} = \mathrm{d}x \ \mathrm{d}y|_{X_f}.$$

Asymmetric logarithmic singularity

Definition. *The roof f has an* asymmetric logarithmic singularity at the origin *if:*

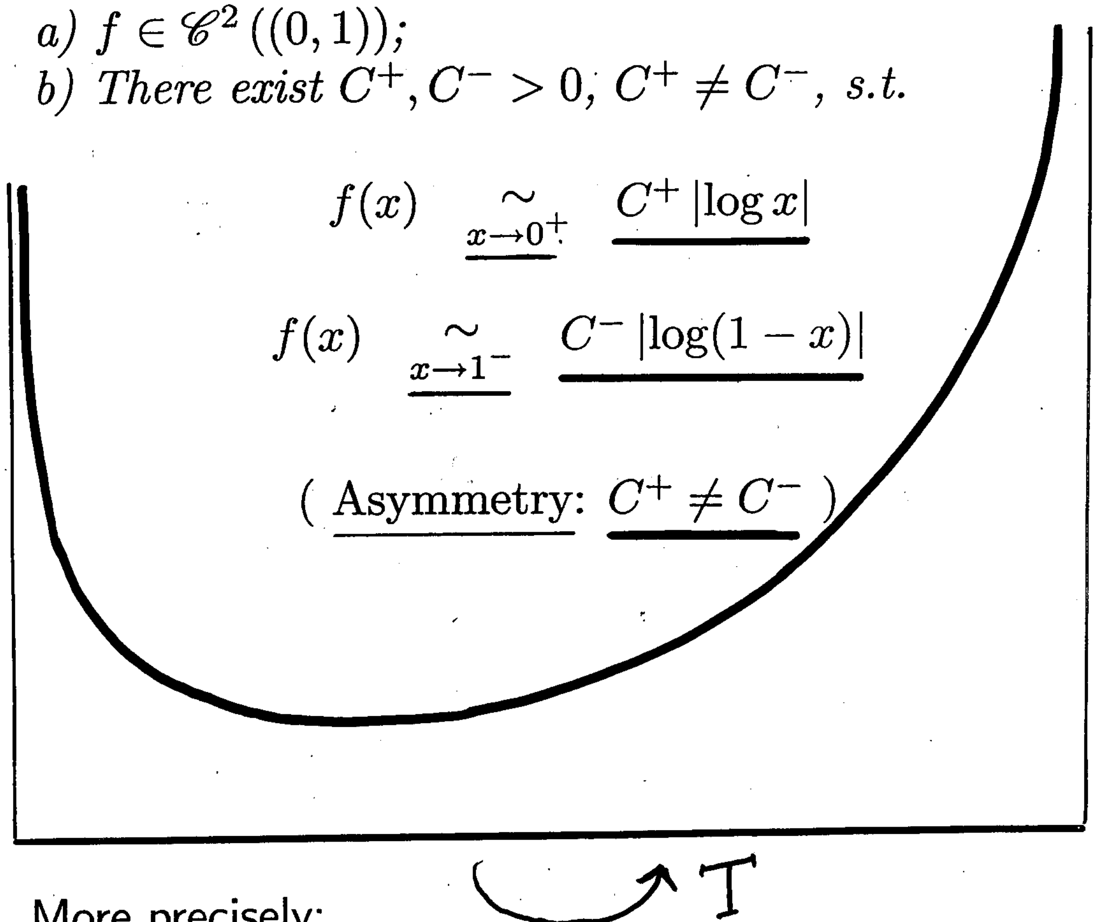
a) $f \in \mathscr{C}^2((0,1))$;

b) There exist $C^+, C^- > 0$, $C^+ \neq C^-$, *s.t.*

$$f(x) \underset{x \to 0^+}{\sim} C^+ |\log x|$$

$$f(x) \underset{x \to 1^-}{\sim} C^- |\log(1-x)|$$

$$(\text{ Asymmetry: } C^+ \neq C^- \text{ })$$



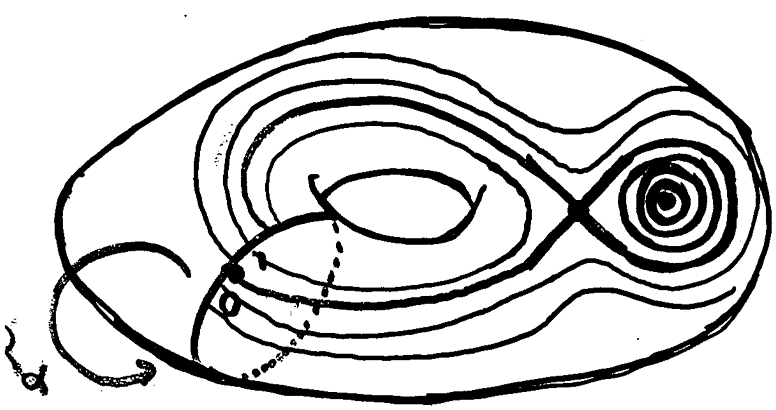
More precisely:

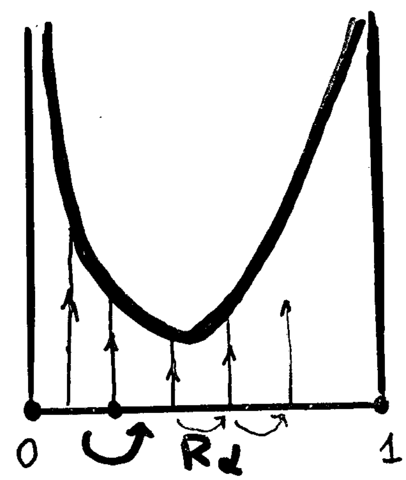
$$\lim_{x \to 0^+} \frac{f''(x)}{\frac{1}{x^2}} = C^+; \qquad \lim_{x \to 1^-} \frac{f''(x)}{\frac{1}{(1-x)^2}} = C^-.$$

Reduction to suspension flows

E.g.: Consider the torus $\mathbb{T}^2$, ω Morse closed 1-form; (e.g. 1 saddle). Let T be $R_\alpha : x \mapsto x + \alpha \pmod 1$

Claim: The flow on the minimal component is metrically isomorphic to a suspension flow φ_t over R_α under a roof with a logarithmic asymmetric singularity.

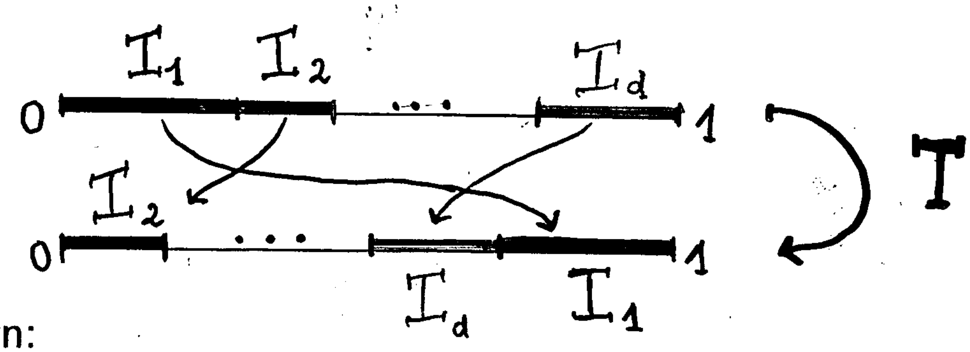




Poincaré cross section	$\longleftrightarrow$	rotation R_α
non-degenerate saddle	$\longleftrightarrow$	logarithmic singularity
presence of the island	$\longleftrightarrow$	asymmetry

Interval Exchange Transformations

- $T : [0,1] \to [0,1]$; decompose I as $I_1 \cup \cdots \cup I_d$;



- Assign:

  a) a lengths vector: $\lambda_i = |I_i|$ length of I_i;
  $\underline{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d_+$, $\lambda_i > 0$, $\sum_{i=1}^d \lambda_i = 1$;

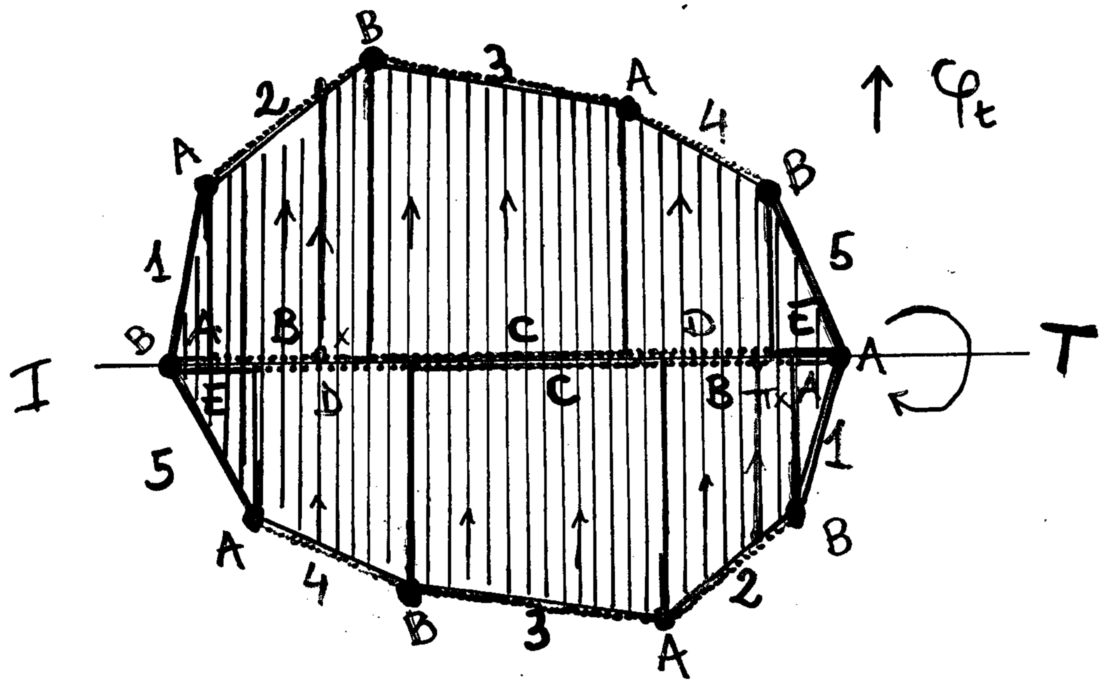
  b) a permutation $\pi \in \mathcal{S}_d$, $\{1, \ldots, d\} \xrightarrow{\pi} \{1, \ldots, d\}$

- New order: $I_{\pi^{-1}1}, \ldots, I_{\pi^{-1}d}$;
  if $x \in I_j$, $T(x) = x + \delta_j$; $(\delta_j = \sum_{i=1}^{j-1} \lambda_{\pi^{-1}i} - \sum_{i=1}^{j-1} \lambda_i)$

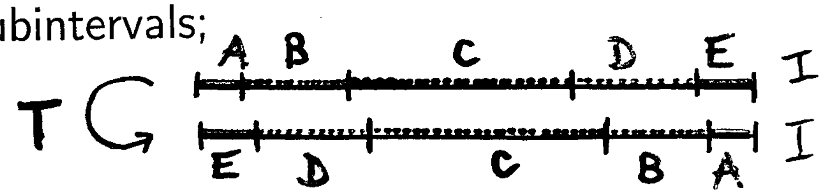
Remark. *T preserves* $\mathrm{d}x$*;*
(T is an orientation-preserving piecewise-isometry)

Interest:
- Generalization of rotations ($d = 2$);
- Poincaré first return maps of area and orientation preserving flows on surfaces of higher genus;

IETs as cross sections



- Identify opposite sides; one gets a surface of $g = 2$;
- φ_t flow along vertical lines; it has two simple saddles after identifications;
- the first return map on the horizontal is an IET of $d = 5$ subintervals;

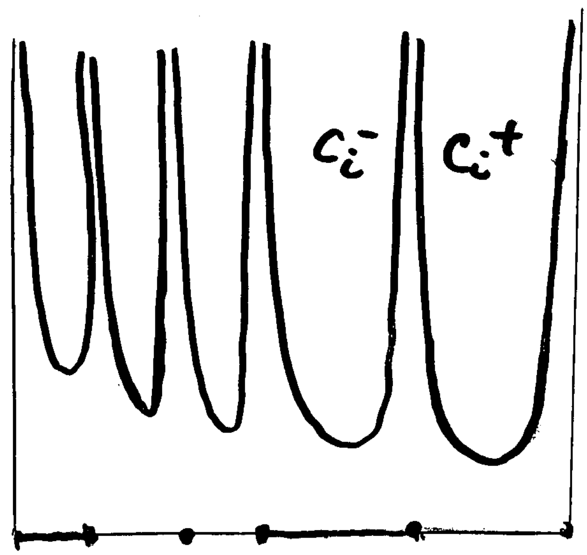


Multi-valued Hamiltonian flows

ω *Morse* closed 1-form on S of genus $g \geq 2$;

Each minimal component is isomorphic to a suspension flow over T:

singularities at discontinuities of T; extra singularities if there are saddle loops:

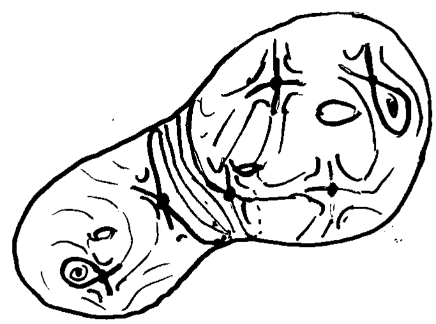


$$f(x) \underset{x \to x_i^{\pm}}{\sim} C_i^{\pm} \left|\log(x - x_i)\right|$$

More generally:

Mixing under the asymmetry condition

$$\sum C_i^+ \neq \sum C_i^-$$

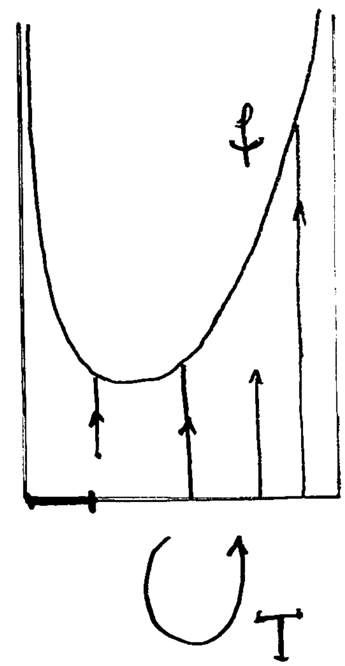


Remark. *The presence of saddle loops is essential to achieve asymmetry.*

Main Results: asymmetric case

Theorem 1. *For a full measure set of interval exchange transformations* $T = (\pi, \lambda)$ *the suspension flow* $\{\varphi_t\}_{t\in\mathbb{R}}$ *over* T *under a roof having an asymmetric logarithmic singularity is* mixing.

- *Full measure* means: for every irreducible π (i.e. $\pi\{1,\ldots,k\} = \{1,\ldots,k\}$ iff $k = d$) for Lebesgue-a.e. lengths-vector λ;



(*Ref:* Ergodic Theory Dynam. Systems)

- *Tools:*

  - Rauzy-Veech renormalization algorithm (generalization of continued fraction);
  - Recent result by Avila-Gouezel-Yoccoz (for full measure);

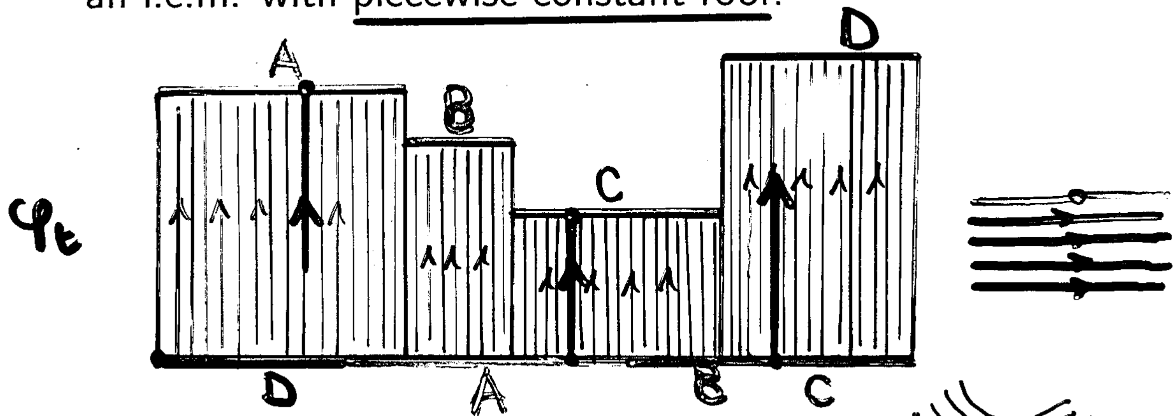
- *Ingredients:*

  - Diophantine-type of condition for IETs;
  - Growth of Birkhoff sums of $\frac{1}{x}$;

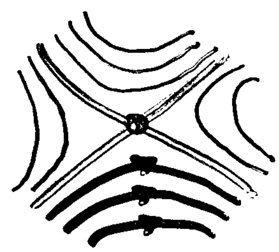
Reduction to Suspension Flows

To represent the flow, use *suspension flows over i.e.m.*

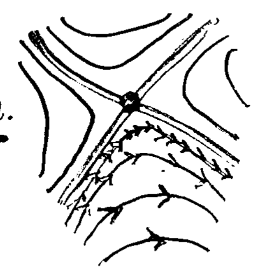
- *e.g. :* Zippered rectangles: vertical geodesic flow on a translation surface → suspension flow φ_t over an i.e.m. with piecewise-constant roof.



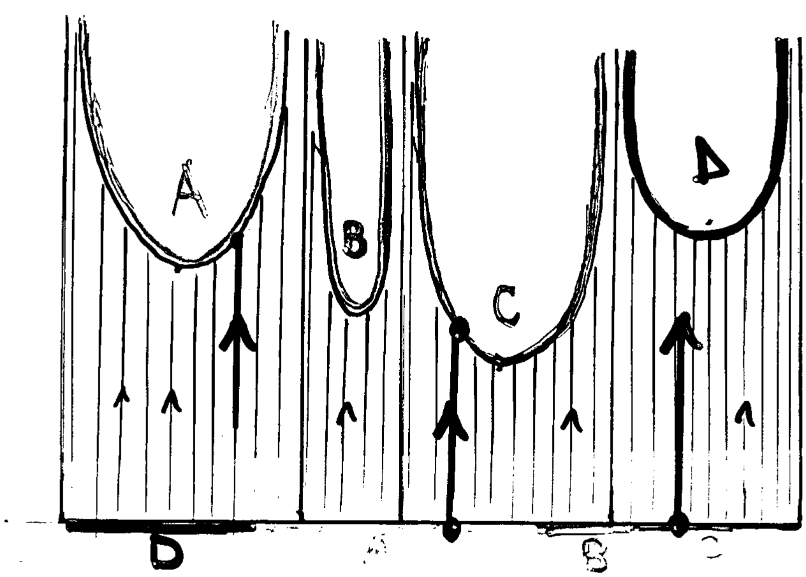
Recall: this φ_t is never *mixing* (Katok); for $g \geq 2$, φ_t is typically *weakly mixing* (Avila-Forni);



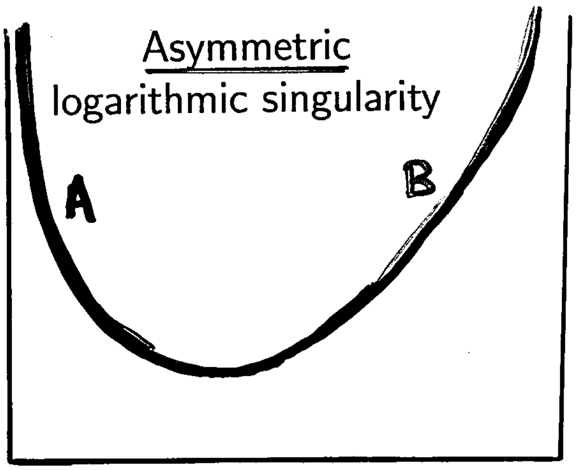
- Hamiltonian saddles "slow down trajectories" → logarithmic singularities of the roof;

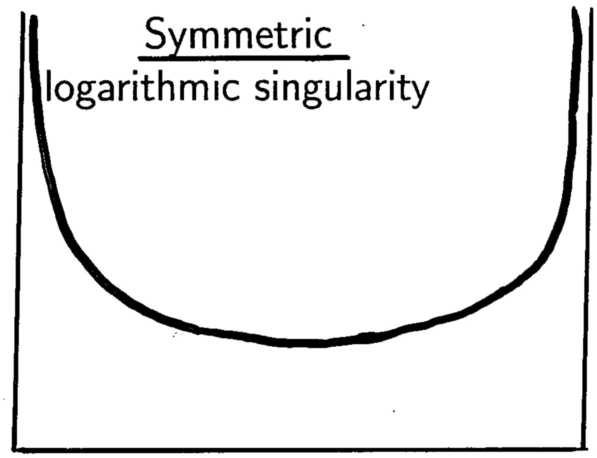


Remark. *Mixing depends on the parametrization.*



Models of logarithmic singularity

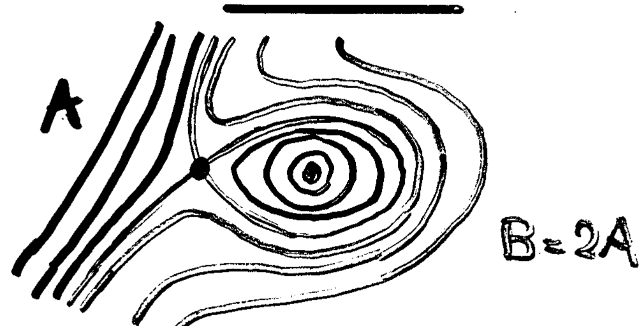




$F_1 = A|\ln x| + B|\ln(1-x)|$, $A \neq B$ $\qquad$ $F_2 = |\ln x| + |\ln(1-x)|$

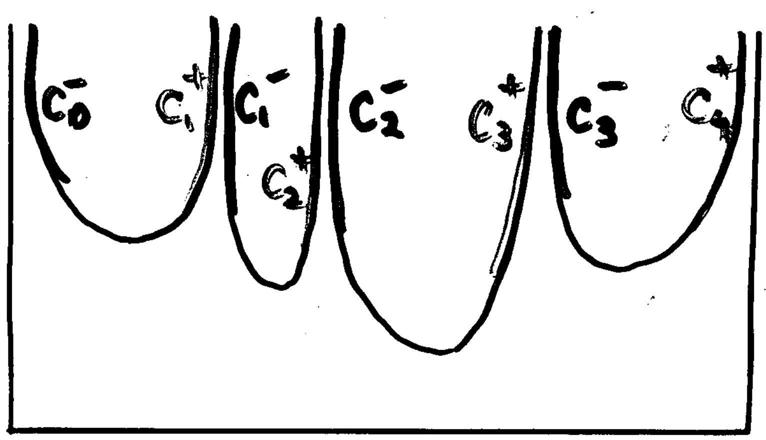
$F_1 \in LogAsym$ $\qquad$ $F_2 \in LogSym$

Remark: Centers (or saddle loops) create asymmetry:



More generally:

$f(x) \underset{x \to x_i^\pm}{\sim} C_i^\pm \left|\log(x - x_i)\right|,$ $\qquad$ $\sum C_i^+ \neq \sum C_i^-$



Main results: symmetric case

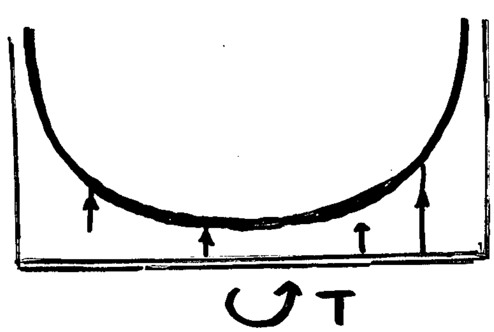
Theorem 2. *Let φ_t be a suspension flow over an interval exchange transformation T under f with a logarithmic symmetric singularity.*

For a full measure set of interval exchange transformations, φ_t is weakly mixing.

- *Full measure*: for every irreducible π (i.e. $\pi\{1,\ldots,k\}=\{1,\ldots,k\}$ iff $k=d$) for Lebesgue-a.e. lengths-vector $\underline{\lambda}$;

Theorem 3. *For a special class of IETs ("bounded type") the suspension flow φ_t over T under f with a logarithmic symmetric singularity is* not mixing.

- *Remark:* In particular, this gives examples of flows φ_t which are weakly mixing, but not mixing.



Renormalization Algorithm

The *Rauzy-Veech algorithm* produces a sequence:

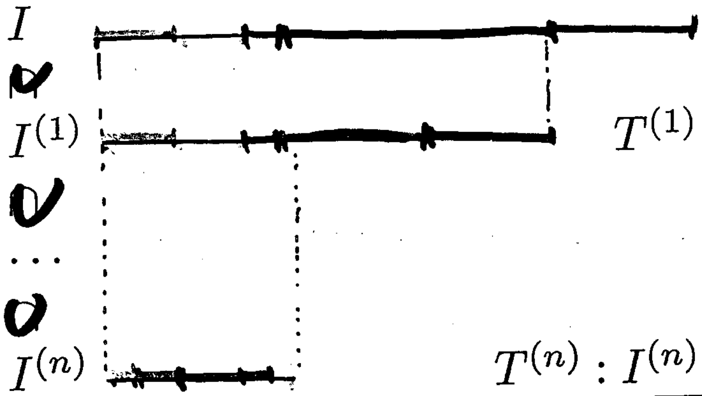
I $\quad T : I \to I$

$I^{(1)}$ $\quad T^{(1)} : I^{(1)} \to I^{(1)}$

...

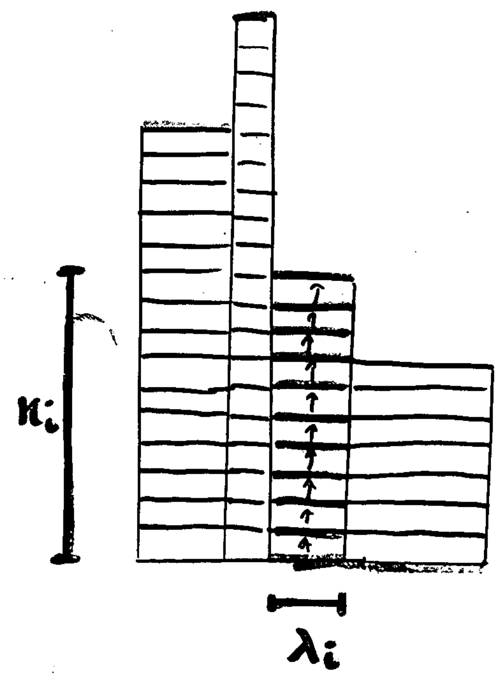
$I^{(n)}$ $\quad T^{(n)} : I^{(n)} \to I^{(n)}$

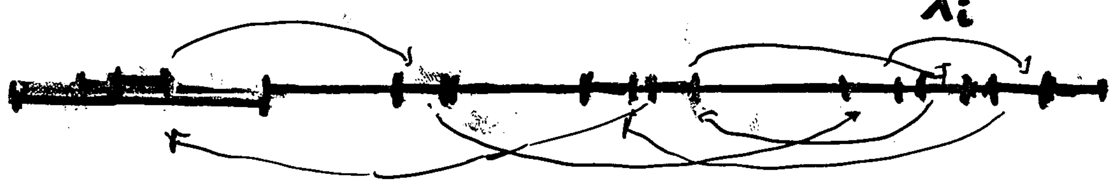
...



$T^{(n)}$ induced map on $I^{(n)}$, IET of $I_1^{(n)}, \ldots, I_d^{(n)}$;

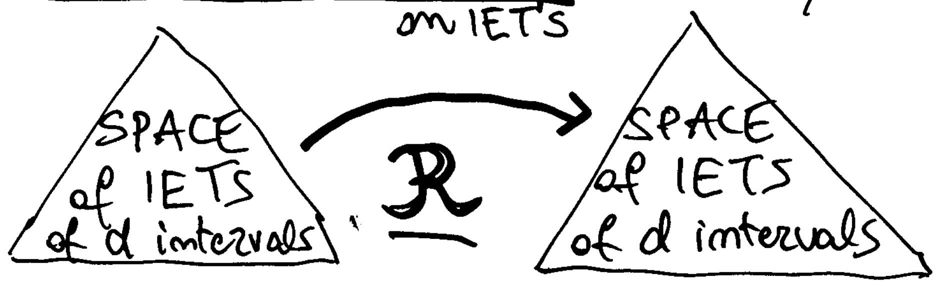
- $\lambda_i^{(n)} \doteqdot |I_i^{(n)}|$ new lengths;
- $\underline{r}^{(n)} \doteqdot (r_1^{(n)}, \ldots, r_d^{(n)})$ return times, i.e. $r_i^{(n)} \doteqdot \min\{r |\ T^r I_i^{(n)} \subset I^{(n)}\}$;
- $\underline{r}^{(n)} = A^{(n)} \cdot \underline{r}^{(n-1)}$, $A^{(n)} \in SL(d, \mathbb{Z})$; (Rauzy-Veech cocycle)

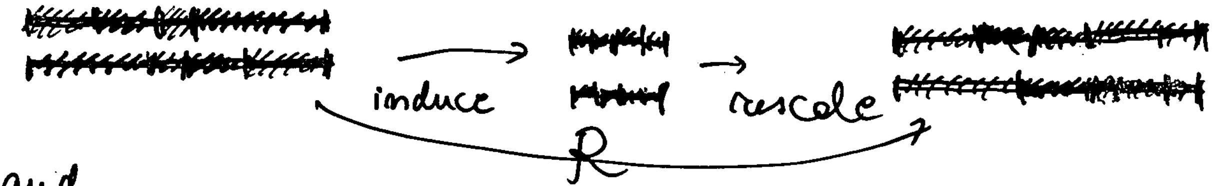




Renormalization and Teichmüller flow

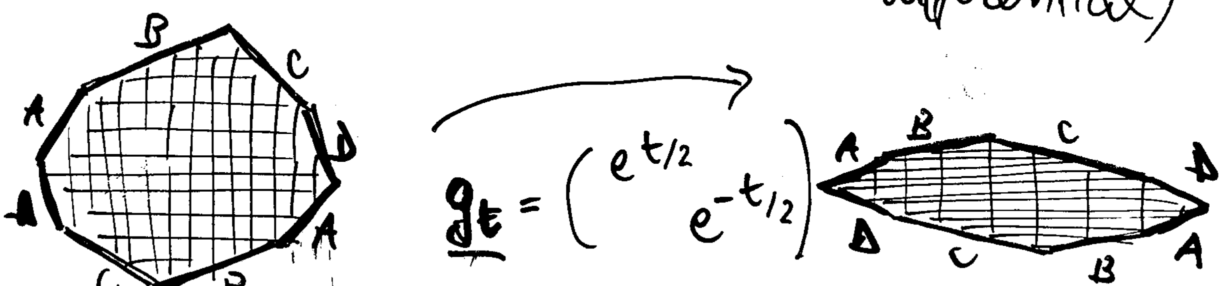
1) • **Renormalization map** on IETS (Rauzy-Veech)





2) and • **Teichmüller geodesic flow** on the MODULI **space of FLAT SURFACES**

((X, ω) Riemann surfaces + Abelian differential)



$$g_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$$

quotient the Mod(g) action

• are related: g_t is (up to a finite cover) a **SUSPENSION FLOW over the renormalization map** $\mathcal{R}$.