Model Manifolds for Surface Groups

Talk by Jeff Brock

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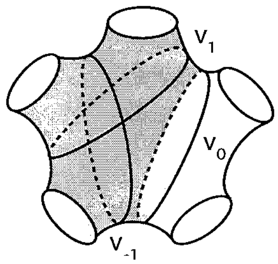
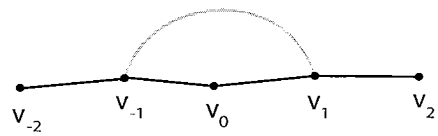
One of the themes of this course has been to emphasize how the combinatorial structure of Teichmüller space can be used to understand hyperbolic manifolds. An important tool throughout has been the observation that knowing the bounded length curves on a surface is almost as good as knowing its exact position in Teichmüller space. As an example, given any curve α and transversal β on a punctured torus, the subset of Teichmüller space consisting of surfaces on which both α and β are bounded is a compact set. So while the names of the bounded length curves on a surface do not specify a its position Teichmüller space exactly, it does do so up to bounded error. The same is true in higher genus for a bounded length pants decomposition together with transversals.

In some cases coarse information is all we need. As Minsky showed in a previous lecture, one can use coarse information about the ends of a quasi-Fuchsian manifold to construct a coarse model of the manifold. Given (ν^-, ν^+), a pair of conformal boundaries or ending laminations, we saw how to associate a manifold M_{ν^-,ν^+} given by gluing blocks according to combinatorial rules given by the ending data. While this model is not exact, in some cases it is precise enough to uniquely specify a manifold. For example, the bi-Lipschitz rigidity of hyperbolic manifolds whose limit sets are the entire Riemann sphere gives that two hyperbolic manifolds with the same models must be isometric, which gives a proof of the ending lamination conjecture in this case. This shows that the coarse perspective can be very powerful.

The case of the 5-Holed Sphere-

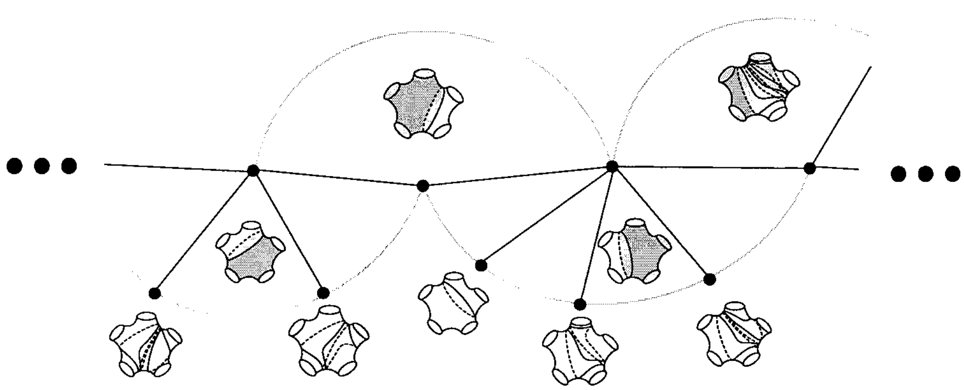
In this section we will discuss how to construct the model manifold for 5-holed sphere groups. This is the simplest a case in which we can see how points from lower complexity curve complexes fit together into something called a hierarchy, and how information from the hierarchy can be used to construct a model. The following picture illustrates a geodesic in the curve complex of the 5-holed sphere.





Notice that each curve v on the 5-holed sphere cuts the surface into a 4-holed sphere component and a pair of pants. The curves at distance two along a geodesic, for example v_{-1} and v_1 in the illustration above both sit in the four holed sphere component cut off by the curve v_0 between them on the curve complex geodesic. We would like to understand how v_{-1} and v_1 are related in the curve complex of the component of $S - v_0$ homeomorphic to a 4-holed sphere. In the case shown above, v_{-1} and v_1 intersect exactly twice, which is the minimal number of times two distinct essential curves can intersect in the 4 holed sphere. Thus v_{-1} and v_1 are at distance 1 in the curve complex of the subsurface. This is indicated by a gray edge joining these two points in the above illustration. Note that this edge is not an edge in the curve complex of the 5 holed sphere, but is an edge in the lower complexity subsurface.

Given any two points v_{i-1} and v_{i+1} that are at distance two in the curve complex, we can attach extra edges to show how they are related in the curve complex of the 4-holed sphere cut off by the curve v_i between them. In the above illustration, v_1 and v_{-1} are adjacent curves in this lower complexity curve comples. In general v_{i-1} and v_{i+1} can be very far apart. Denote by W_{v_i} the component of the complement of v_i that is topologically a 4-holed sphere, and let $g_{v_i} \subset \mathcal{C}(W_{v_i})$ be a geodesic between the points v_{i-1} and v_{i+1}. We associate to the geodesic g all the points along the geodesic g_{v_i}, which gives us a collection of curves called a hierarchy.



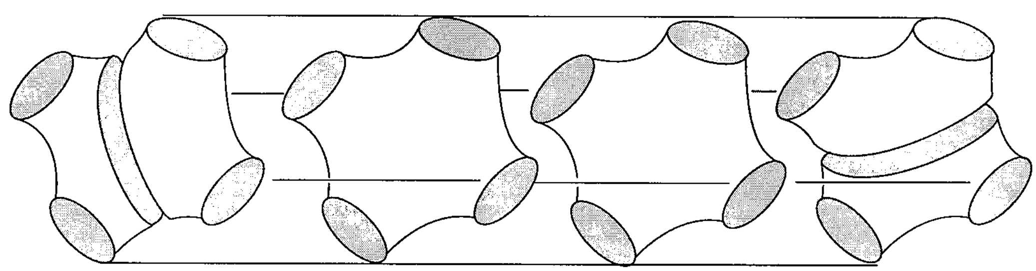
Each black edge in the above picture corresponds to a pants decomposition of the 5-holed sphere, namely the pants decomposition given by its two endpoints. Notice also that these pants decompositions are ordered by their arrangement along the core geodesic of the hierarchy. Given a pair of pants decompositions P_0 and P_1 of $S_{0,5}$, we can build a hierarchy $H(P_0, P_1)$ connecting them by taking any geodesic between them in $\mathcal{C}S_{0,5}$ and then augmenting it as above. This construction is due to Masur and Minsky, and the collection of curves in the hierarchy serve as the analog of the pivot sequence in the puncture torus case. The following theorem gives that just like the curves in the pivot sequence, the curves in the hierarchy have bounded length in the manifold.

THEOREM: (Minsky) Given $X, Y \in Teich(S_{0,5})$ with pants decompositions P_X and P_Y each of length bounded by the Bers constant on X and Y respectively, then there exists L such that for any $v \in H(P_X, P_Y)$, $l_{Q(X,Y)}(v) < L$.

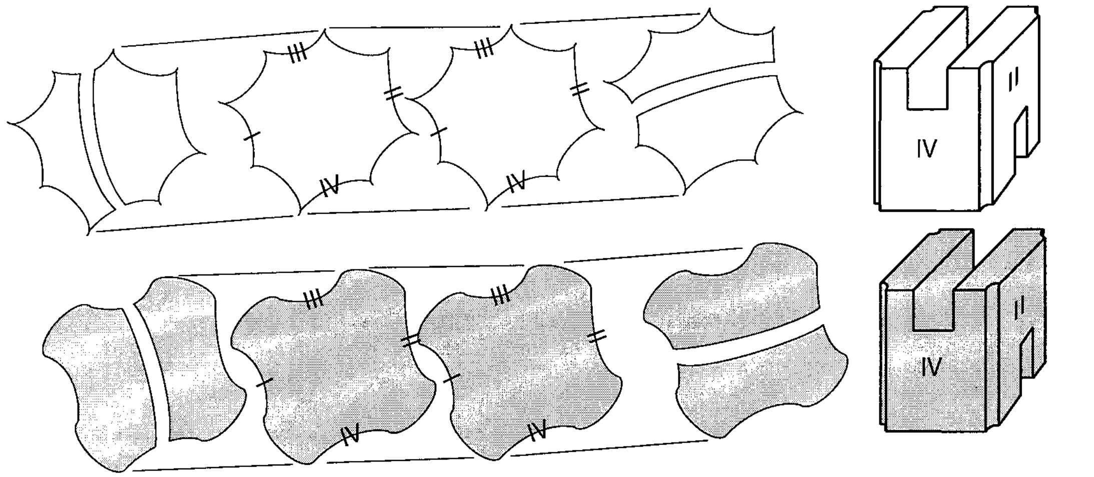
The above theorem is sometimes referred to as the "a priori bounds" theorem.

BUILDING THE MODEL-

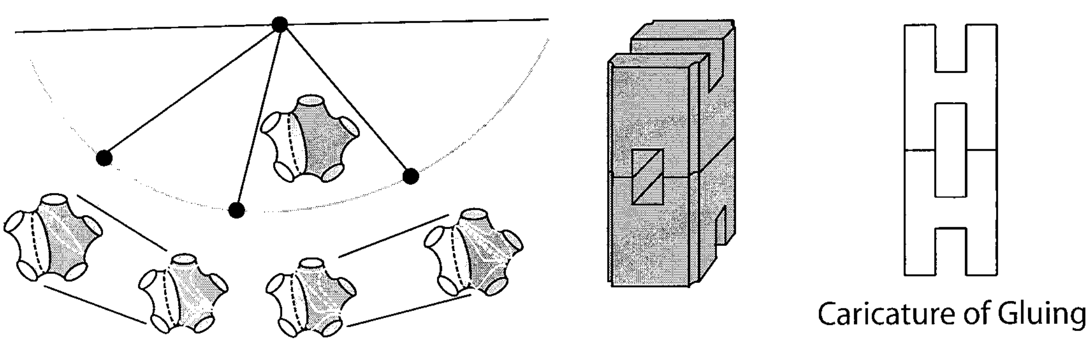
Given $X, Y \in Teich(S_{0,5}$ and bounded length pants decompositions P_X and P_Y, we can a manifold M_{XY} and a Lipschitz map $f : M_{XY} \to Q(X,Y)$ using the curves in $H(P_X, P_Y)$. While we will not describe the details of the map f, we will show how the model is constructed. The elementary pieces of M_{XY} are blocks cut from $S_{0,4} \times [0,1]$, and we have one such block for each of the gray edge in picture of the hierarchy above. For an edge in the curve complex of the four times punctured sphere W_v whose endpoints correspond to the curves γ_1 and γ_2, we have the block $W_v \times [0,1] - \mathcal{N}_\epsilon(\gamma_1) \times [0,\epsilon] \cup \mathcal{N}_\epsilon(\gamma_1) \times [1-\epsilon, 1]$.



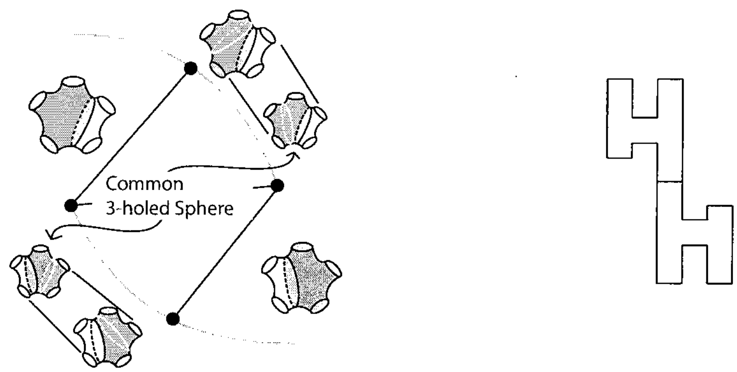
Cutting the 4-times punctured spheres into their bottom and top halves, we can exhibit this block as a gluing construction.



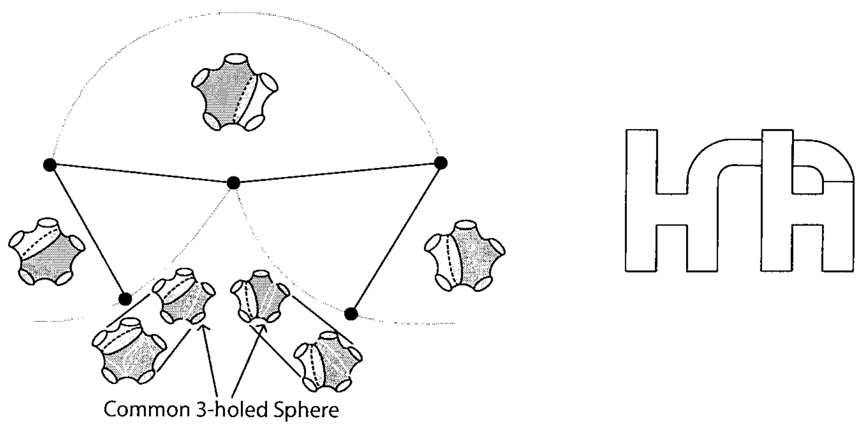
We now describe how to glue these blocks together. There are three types of gluings, depending on the relative positions of the blocks. The first type of gluing occurs when both blocks are cut from the same 4-holed sphere, in which case the 3-holed spheres on facing ends of the blocks are identical.



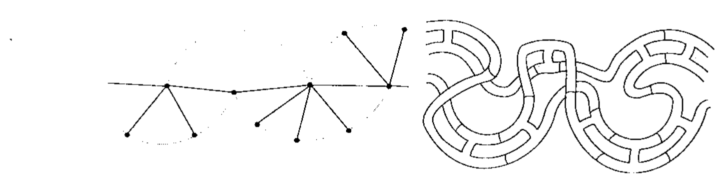
The second kind of gluing occurs between edges on opposite sides of the core geodesic of the hierarchy. In this case one 3-holed sphere component from each block is glued together.



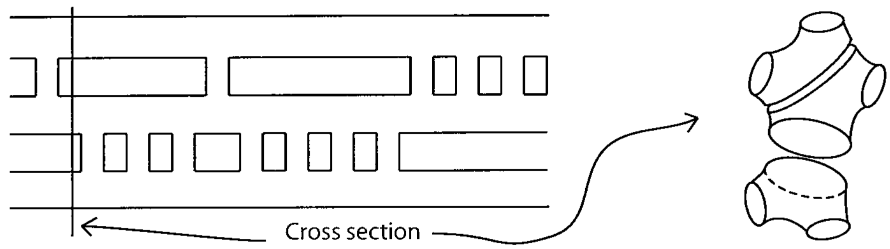
The final type of gluing occurs when the trough at the end of each block corresponds to the same curve, but the blocks are cut from different 4-holed spheres.



These gluings give a chain of blocks whose arrangement is dictated by the hierarchy:



Untwisting the above picture, we see an arrangement like this:



After gluing solid tori with the appropriate geometry into each of the holes in the above picture, we get a manifold that is topologically $S_{0,5} \times \mathbb{R}$, which can be shown to be a Lipschitz model for the original manifold.

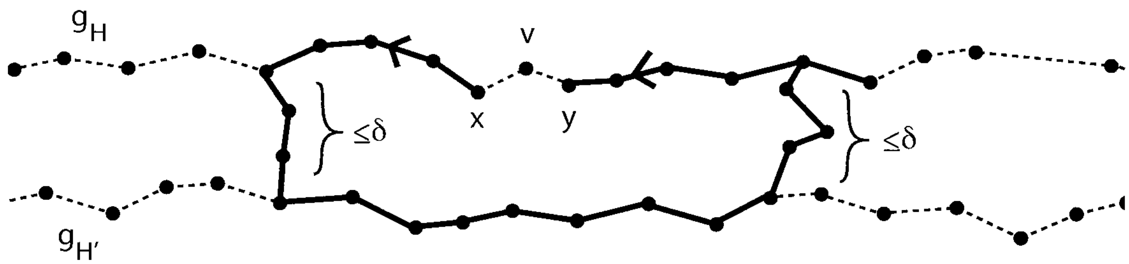
"UNIQUENESS" OF THE MODEL-

In the punctured torus case, the model manifold was strictly determined by the positions of the points X and Y in Teichmüller space. The only curves that were used in the construction of the model came from edges in the Farey graph that had to be crossed to get between X and Y. Hierarchies, on the other hand, cannot be chosen canonically so one might wonder why two models based on different hierarchies should have anything to do with one another. As was mentioned earlier, it is the δ-hyperbolicity of the curve complex that saves the day.

PROPOSITION: Let H and H' be two hierarchies joining P_X and P_Y, g_H and $g_{H'}$ their core geodesics. There exists a constant K such that if $v \in g_H$, $v \notin g'_H$ and x and y are its neighbors along g_H, then $d_{W_v}(x,y) < K$.

This shows that if v shows up in one hierarchy and not in another, then the wheel attached to v cannot be too long. Applying the contrapositive of the proposition, if we see a very large wheel around v in some hierarchy, then v has to show up in every hierarchy. These facts can be used to show that any two models for the same two pair of points in Teichmüller space are bi-Lipschitz equivalent.

Proof. The idea of the proof is to create a short path from x to y that does not contain v, and then use subsurface projection to make this into a short path in $\mathcal{C}(W_v)$. If we walk along g_H in either direction from v, we do not see the curve v again as g_H is a geodesic. By δ-hyperbolicity, the geodesics g_H and $g_{H'}$ are fellow travelers in the curve complex, so from any point along g_H we can walk to the corresponding point along $g_{H'}$ using δ steps. If we walk at least a distance δ away from v, then we can ensure that we don't hit v when we walk to the other geodesic in less than δ steps. If we do this both in the direction of x and in the direction of y, then we can cross over to two points on $g_{H'}$ that are a distance of 2δ apart.



By assumption v is not on the geodesic $g_{H'}$, so we can connect these two points on $g_{H'}$ without hitting v. This gives us a path $x = p_0, p_1, ..., p_m = y$ in the curve complex from x to y that avoids v and is of length less than 4δ.

We now use these points to get a path in $\mathcal{C}(W_v)$. p_i and p_{i+1} are disjoint curves, and it is easy to check that the projection π to the subsurface W_v increases their intersection by at most 2. Curves that intersect twice in W_v are at distance 1, so $d_{W_v}(\pi(p_i), \pi(p_{i+1})) \leq 1$. This shows that $\pi(p_0), \pi(p_1), ..., \pi(p_m)$ is a path in $\mathcal{C}(W_v)$, thus $d_{W_v}(x, y) < 4\delta$.

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