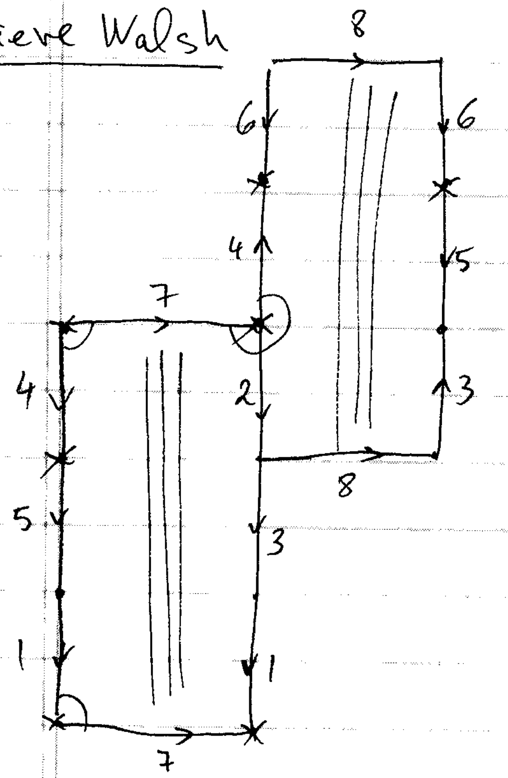
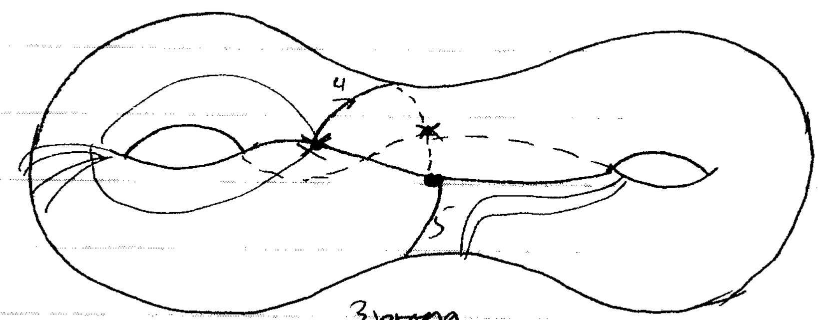
Genevieve Walsh





3 prong singularities

Classification of surface homeomorphisms

→ classifying elements of Mod(S) ; S = surface, orientable, genus $g \geq 1$ (usually $g \geq 2$) no bdry, connected

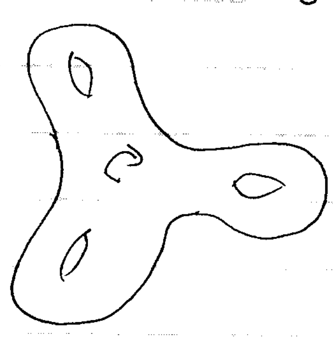
→ $\mathrm{Homeo}^+(S)$ = orientation preserving homeo

→ $\mathrm{Homeo}_0(S)$ = those orient. pres. homeo which are isotopic to the identity

Definition. $\mathrm{Mod}(S) := \dfrac{\mathrm{Homeo}^+(S)}{\mathrm{Homeo}_0(S)}$

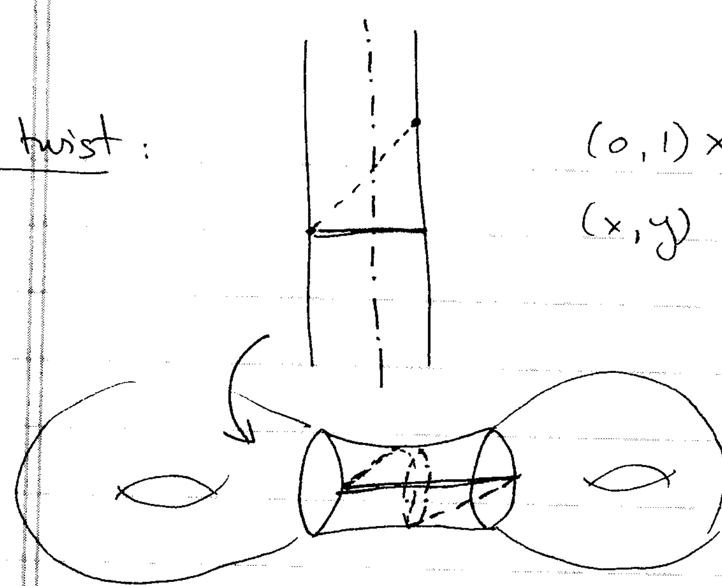
Examples:

(1)



periodic elements of Mod(S) (in picture have per. 3)

2) Dehn twist:



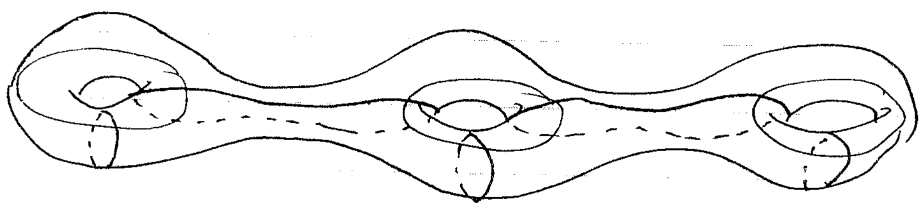
$(0,1)\times\mathbb{R}$

$(x,y) \longmapsto (x, y+2\pi x)$

core of annulus $= \alpha$

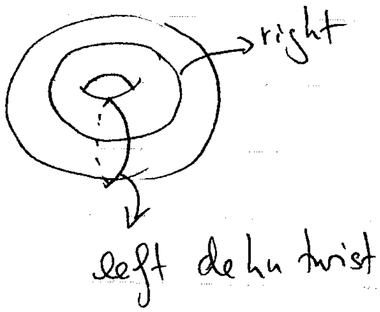
T_α

Dehn (1922) Mod(S) is generated by (finitely many) Dehn twists. $3g-1$ generators



Teruaki www.math.meiji.ac.jp/~ahora/teruaki.html

(3)



$T_b T_a^{-1}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$!

two eigenvectors $\left(\frac{1\pm\sqrt{5}}{2}, 1\right)$

eigenvalues $\frac{3\pm\sqrt{5}}{2}$

2 irrational directions left invariant by this map. on the cover → projects to two irrational foliations left invariant. In the case of the torus they are called Anosov.

Teichmüller space S_g, $g \geq 2$

S $\quad \chi(S) < 0$ $\quad$ X surface w/ hyperbolic metric

Euler characteristic $\quad f : S \to X$ homeomorphism

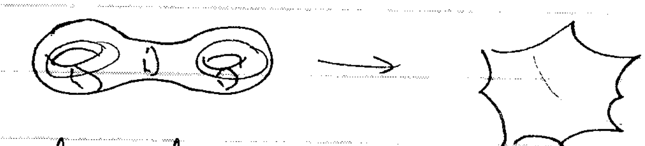
$\mathrm{Teich}(S) = (X, f)/\sim$ where

$(X_1, f_1) \sim (X_2, f_2)$ if $\exists$ isometry s.t. diagram commutes up to homotopy

$$\begin{tikzcd} S \arrow[r, "f_1"] \arrow[dr, "f_2"'] & X_1 \arrow[d, "\text{isom}"] \\ & X_2 \end{tikzcd}$$

i.e. $f_2 \circ f_1^{-1} \sim$ isometry.

$l_\gamma : \mathrm{Teich} \to \mathbb{R}$

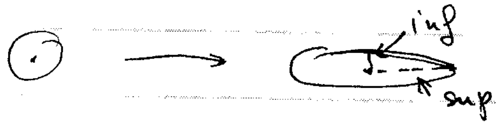


$l(X, f) =$ lenght of geod isot. to γ

$h : X \to Y$ homeomorphism

dilatation of h at p is

$$K_p(h) = \lim_{r \to 0} \frac{\sup_{q \in S_p'(r)} d(h(p), h(q))}{\inf_{q \in S_p'(r)} d(h(p), h(q))}$$



tells how far you are from being conformal

let $K(h) = \sup_{p \in X} K_p(h)$. If finite, h is quasi-conformal

Teich distance (X, f) (Y, g)

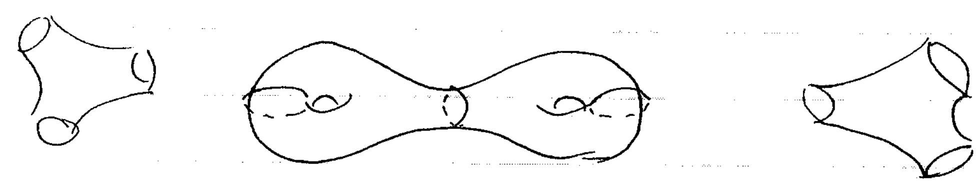
$$d_{\mathrm{Teich}}((X,f),(Y,g)) = \mathrm{Inf}\{\log K(h) : h \sim g \circ f^{-1} \text{ is a quasiconf homeo}\}$$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr] & X \arrow[d, "h \sim g\circ f^{-1}"] \\ & Y \end{tikzcd}$$

For $g=1$, the Teich space is the space of marked unit area lattices.

$\text{Teich}(S) \cong \mathbb{R}^{6g-6}$

Divide every surf into pairs of pants



hyperb structures on pairs of pants are determ by lengths.

The coordinates are $(\ell_1, \dots, \ell_{3g-3}, \theta_1, \dots, \theta_{3g-3})$

twists on gluing.

Wolpert area form: $\omega = \sum d\ell_i \wedge d\theta_i$ does not depend on choice

<u>Back to Mod(S)</u>. Mod(S) acts on Teich space by changing the marking:

$$h(X,f) = (X, f\circ h^{-1}) \qquad \overset{h^{-1}}{\circlearrowleft} S \xrightarrow{f} X$$

mapping class

$$\Rightarrow M(S) = \text{Teich}(S)/\text{Mod}(S)$$

Facts about this action. • <u>Action is by isometries</u>

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "f'"'] & X \arrow[d, "h"] \\ & X' \end{tikzcd}$$

h is K quasi conformal
$h\circ f = f'$ in the right homotopy class

$g \in Mod(S)$ $\quad h \circ (f \circ g^{-1}) \sim f' \circ g^{-1}$

minimal dilatation map has the same K.

- Action is properly discontinuous

Γ is acting on a space; K compact subsp of sp.

$\{ \gamma \in \Gamma : \gamma K \cap K \neq 0 \}$ is finite $\Rightarrow$

$M(S)$ is an orbifold.

$M_\varepsilon(S)$ = the ε-thick part of moduli space

$\{ X \in M(S) \mid \ell(X) \geq \varepsilon \}$

length of shortest curve.

Mumford showed that these sets $M_\varepsilon(S)$ are compact.
S closed surface of genus ≥ 2

Measured foliation of a surface S

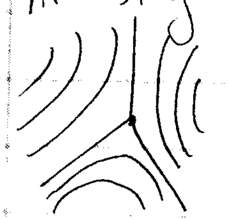
Decompos of S into 1 dimension submanifolds
+ a finite set of singular pts.

non-singular point

transition maps are

y is const $\quad (x, y) \mapsto (f(x,y), y + c)$

At sing point



p-prong singularity

$p=3$

A measured foliation $(\mathcal{F}, \mu)$ ← transverse measure
(singular) foliation ↗

Ex: Foliation of T^2 are either rational or irrational.

xy structure

two obvious foliations

$(\mathcal{F}_x, |dy|)$ and $(\mathcal{F}_y, |dx|)$

xy structure - Identify finite # of rectangles via isom of the bdry

~ surface w/2 transverse measured foliations

Gauss - Bonnet : $\sum D(p) = -2\pi \chi(S)$

Ex: Angle defect is π at each vertex

$D(p) = A(p) - 2\pi$

Neilson-Thurston classification of mapping classes

- Periodic (of finite order)
- reducible

  fixes a collection of distinct disjoint isotopy classes of simple closed curves
- f is pseudo-Anosov, i.e.

  $\exists$ a representative homeomorphism φ and a pair of transverse measured foliations

  $\varphi(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda \mu_u)$

  $\varphi(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1} \mu_s)$

Part II

(X, f) pt of Teich(S) ; equip X with structure

move in Teich space by stretching in horiz. dir. & contracting in vertical direction.

Def: Such a map is a Teichmüller map.

Teich's existence theorem: S closed, $((X,f),(Y,g))$ then $\exists$ a Teich map between (X,f) & (Y,g).

Teich's uniqueness theorem: If $g: X \to Y$ is a Teich map & $g' \sim g$ then $K(g') \geq K(g)$ (or $\exists$ conf map) with equality iff $g' = g$.

Corollary: Teich maps generate geodesics in Teich space.

Proof:

Let $g \in$ Isom (Teich(S)) $\quad \tau(g) = \inf\limits_{X \in \text{Teich}(S)} d(X, gX)$

3 types of isometries:

1. $\tau(g)$ is not achieved : parabolic.
2. $\tau(g)$ is achieved $=0$ elliptic.
3. $\tau(g)$ is achieved >0 hyperbolic

Facts about teich space: • $\exists$ constant $\delta = \delta(S)$ s.t. for $X \in$ Teich any 2 closed geodesic of length $< \delta$ on X are disjoint

• For compact $A \in$ Teich(S), $\exists K$ s.t. $\forall X, Y \in A$

$$\frac{1}{K} \leq \frac{l_X(C)}{l_Y(C)} \leq K \quad , \forall \text{ curve } C \in S$$

Case 1: assume $\tau(f)$ is not achieved. Let $\{X_n\} \subset$ Teich(S) s.t. $d(X_n, f X_n) \Longrightarrow \tau(f)$

Suppose when we project to $M(S)$, stays in comp. set

$\exists g_n$ s.t. $g_n X_n$ stays in a comp. set of Teich(S).

$\Rightarrow \exists$ convergent subseq $\cancel{Y_n} \to Y$.

$$d(Y, g_n f g_n^{-1} Y) \le \overbrace{d(Y, Y_n)}^{\to 0} + d(Y_n, g_n f g_n^{-1} Y_n) +$$
$$+ \underbrace{d(g_n f g_n^{-1} Y_n, g_n f g_n^{-1} Y)}_{0} \quad \text{(b/c isometries)}$$

$$\underbrace{d(Y_n, g_n f g_n^{-1} Y_n)}_{\tau(f)} = d(g_n^{-1} Y_n, f g_n^{-1} Y_n) = d(X_n, f X_n) \to \tau(f)$$

$\tau(f)$ we know

ate

so projections can't stay in a compact set.

$\Rightarrow$ exists every compact set of Moduli space.

$\Rightarrow$ (by Mumford's compactness theorem) the length of some curve $\to 0$.

Let A be ball of radius $\tau(f)+1$. By prev. theor.

$\exists K$ s.t. $\ell_X(c) \le K \ell_Y(c)$ in this ball $(X, Y \in A)$

Take n big enough s.t.

$$d(X_n, f(X_n)) < \tau(f) + 1 \ \& \text{ s.t. } \ell(X_n) < \left(\tfrac{1}{K}\right)^{3g-3} \cdot \delta$$

Let C_0 a shortest curve in X_n. $\ell_{X_n}(f^{-1} C_0) = \ell_{f X_n}(C_0) \le$
$\le K' \ell_{X_n}(C_0) < \delta$

$C_0, f^{-1}(C_0), \ldots, f^{-(3g-3)} C_0$ can't be all distinct

This is a reducing set of curves.

So $\tau(f)$ not achieved $\Rightarrow f$ is reducible.

Case 2 $\tau(f)=0 \Rightarrow \exists X \in \text{Teich}$ s.t. $fX = X$

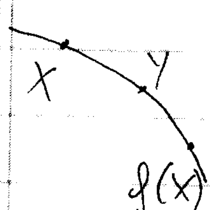
$\Rightarrow$ isometry of X $\Rightarrow$ periodic

$|\text{Isom}(S_g)| \leq \Theta\, 84(g-1)$

Case 3 $\tau(f)$ achieved and positive.

$d(X, f(X)) = \tau(f)$

Claim f fixes the unique geodesic γ connecting X to fX.



let Y be the midpoint on this geodesic

$$\Rightarrow d(Y, fY) \leq d(Y, fX) + d(fX, fY)$$

$$\tfrac{1}{2}\tau(f) + \tfrac{1}{2}\tau(f) = \tau(f)$$

$\Rightarrow$ $f(Y)$ has to be on the geodesic. etc.,

by the same argument $f^i(X)$ lies on same geodesic

$\Rightarrow f(\gamma) = \gamma$.

So you move along a geodesic by Teich. maps.

f takes the measured ~~lamination~~ foliation defining the geodesic γ to the one with the measures changed

$\Rightarrow$ p-Anosov

[See Notes of Farb & Margulit]

Theorem (Thurston)

$$M_\varphi = S \times [0,1] \big/ (X,0) \sim (\varphi(X), 1)$$

If • $[\varphi]$ is periodic $\Rightarrow M_\varphi$ admits a geometric structure of $\mathbb{H}^2 \times \mathbb{R}$

• $[\varphi]$ is reducible $\Rightarrow M_\varphi$ contains an (immersed) incompres. torus.

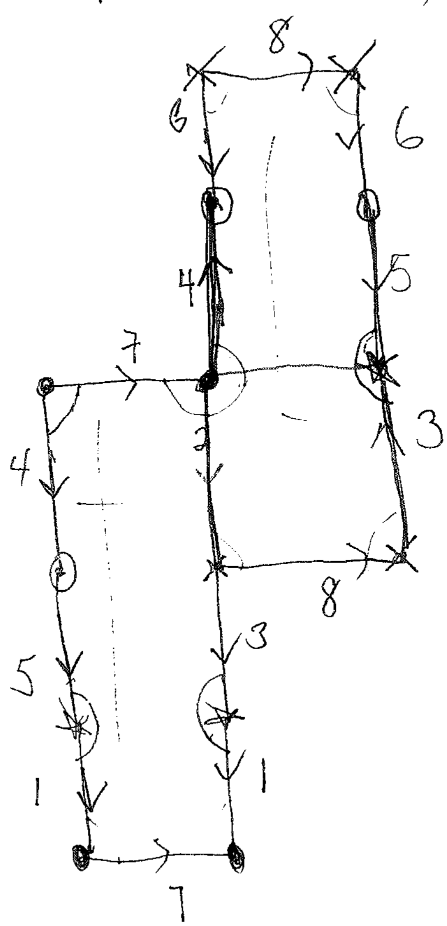
• $[\varphi]$ is p-Anosov $\Rightarrow M_\varphi$ is hyperbolic.

Projective classes of measured foliations give a natural boundary of Teichmüller space.
A parabolic will fix 1 pt. on the bdry, an elliptic will fix some point in the interior.

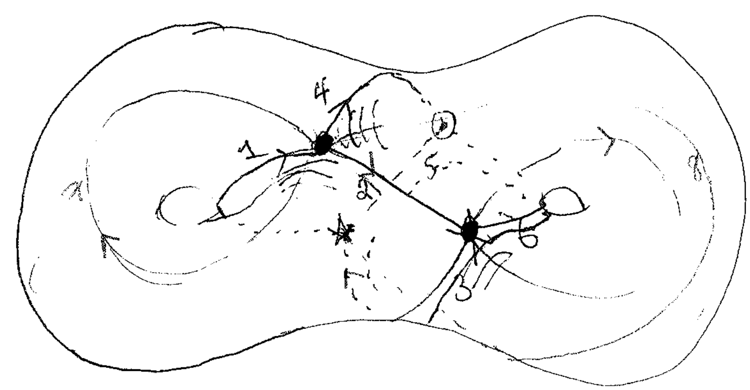
(~conformal structure)

✯ This needs to go on the board.

put picture on board.



vertical foliation.



Ex: think about the horizontal foliations.

Gauss-Bonnet

$\sum D(p) = -2\pi\chi(S)$

regular

$D(p) = A(p) - 2\pi$

✯ π

• π

∘ π

⊛ X π

$\chi(S) = -2$

genus 2.