Benson Farb: www.math.utah.edu/~margalit/Primer

A primer on mapping class group

Problems on mapping class groups and related topics. Proc. Pure math AMS

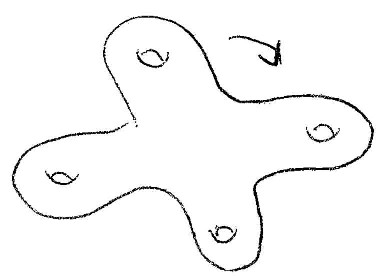
also on Farb's web page

Basics

let S = compact oriented surface

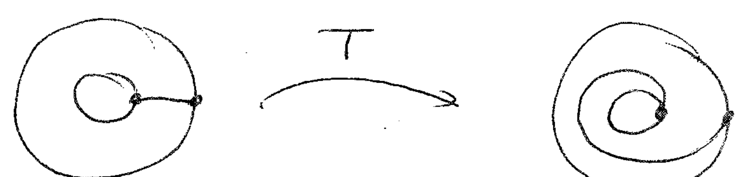
$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S, \partial S)/\mathrm{Homeo}^0(S, \partial S)$$
$$= \pi_0\left(\mathrm{Homeo}^+(S, \partial S)\right)$$
$$= \mathrm{Homeo}^+(S, \partial S)/\sim \qquad f \sim g = f \text{ homotopic } g$$

Ex:

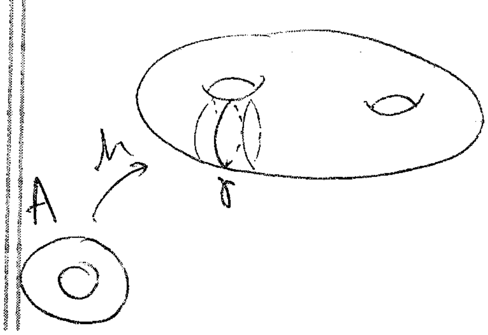


Dehn twists: let $A = \left\{ (r, \theta) \;\middle|\; \begin{array}{l} 0 \le \theta \le 2\pi \\ 1 \le r \le 2 \end{array} \right\}$

let $T(r, \theta) = (r, \theta + 2\pi r)$



Now let $\gamma \subset S$ be a simple closed curve on S (scc),
= embedded $S^1 \to S$



Fix h homeo : $A \to \overline{\mathrm{Nhood}(\gamma)}$

Define $T_\gamma \in \mathrm{Homeo}^+(S)$ via

$$T_\gamma(z) = \begin{cases} h \circ T \circ h^{-1}(z) & \in \overline{\mathrm{Nhood}(\gamma)} \\ z, & \text{if } z \notin \overline{\mathrm{Nhood}(\gamma)} \end{cases}$$

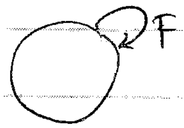
• As an element of $\mathrm{Mod}(S)$, T_γ depends only on the free class of γ.

__Exercise__: Prove T_γ has ∞ order in $\mathrm{Mod}(S)$ if $\gamma \not\simeq *$ (γ is nontrivial)

__Alexander Trick__ (lemma) $\mathrm{Mod}(D^2) = 0$

any homeom. of the closed disc which is the ident on the bdr, it is homotopic to the identity.

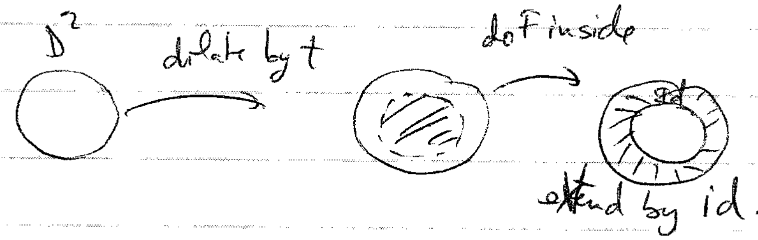
__Proof__:



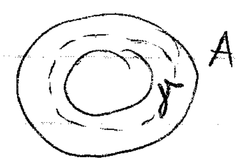
$F \in \mathrm{Homeo}(D^2)$

$F|_{\partial D^2} = \mathrm{Id}.$

$0 \le t \le 1$

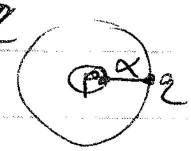


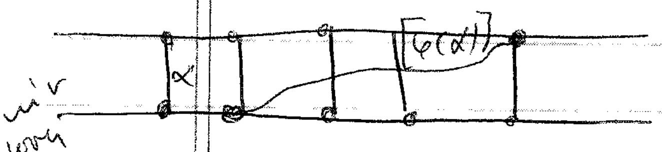
__Proposition__: $\mathrm{Mod}(A) \simeq \mathbb{Z} = \langle T_\gamma \rangle$



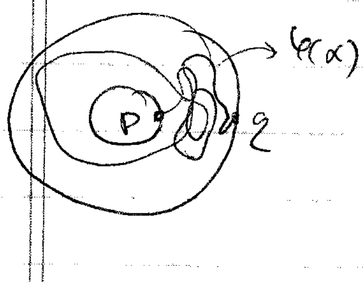
Proof: $\Psi: \mathrm{Mod}(A) \longrightarrow \pi_0(A, p, q) = \mathbb{Z}$

$\varphi \longmapsto [\varphi(\alpha)]$





onto: $\Psi(T_\gamma^n) = n$



1-1: show that if $\varphi(\alpha) \sim \alpha$, $\varphi \sim \mathrm{Id}$

• homotop φ s.t. can assume $\varphi(\alpha) = \alpha$

• apply Alexander trick to "disc" 

φ induces a homeomorph of D^2 ↗.

$\mathrm{Mod}(T^2)$: $\quad A \in SL_2\mathbb{Z} \longrightarrow \varphi_A \in \mathrm{Mod}(T^2)$

$T^2 \simeq \mathbb{R}^2 / \mathbb{Z}^2$

__Theorem__: The natural map $\mathrm{Mod}^{\pm}(T^2) \xrightarrow{\simeq} \mathrm{Out}(\pi_1 T^2) \simeq \mathrm{Out}(\mathbb{Z}^2)$

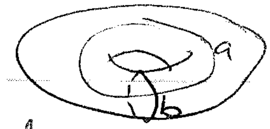
$\varphi \longmapsto \varphi_*$ $\qquad \simeq GL(2, \mathbb{Z})$

includes orientation reversing

is an isomorphism

Inverse map is $A \longmapsto \varphi_A$ (gives onto).

1-1: If $\varphi_* = \mathrm{Id}$. want $\varphi \sim \mathrm{Id}$.



$$\Downarrow$$

$$\varphi(a) \sim a$$

$$\varphi(b) \sim b$$ now quote Alexander □

Corollary: $\mathrm{Mod}(T^2) = \langle T_a, T_b \rangle$

Dehn - Nielsen - Baer

Let $g \geq 2$, let Σ_g be a closed genus g surface. Then the natural map $\mathrm{Mod}^{\pm}(\Sigma_g) \longrightarrow \mathrm{Out}(\pi_1 \Sigma_g)$ is an isomorphism.

Dehn Twist Basics.

let a, b, α scc. on S

1. $a \sim b \iff T_a = T_b$ (in $\mathrm{Mod}(S)$)
2. $\forall f \in \mathrm{Mod}(S)$ $f T_\alpha f^{-1} = T_{f(\alpha)}$.
3. $T_a T_b = T_b T_b \iff i(a,b) = 0$

$i(a,b) =$ intersection number of a & b.

$$i(a,b) = \min_{\substack{a' \sim a \\ b' \sim b}} |a' \cap b'|$$

Proof: "$\Leftarrow$" clear.

"$\Rightarrow$" Assume $T_a T_b = T_b T_a$

$$\iff T_a T_b T_a^{-1} = T_b \overset{②}{\iff} T_{T_a(b)} = T_b \overset{①}{\iff} T_a(b) \sim b$$

But $i(T_a^k(b), b) = |k|\, i(a,b)^2$

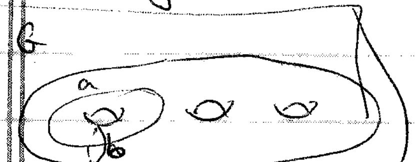
Suppose by contradiction $i(a,b) \neq 0 \Rightarrow i(T_a(b), b) > 0$ so $T_a(b) \sim b$ ⨳.

4. (Braid Relation) $i(a,b) = 1 \Rightarrow T_a T_b T_a = T_b T_a T_b$

Proof:

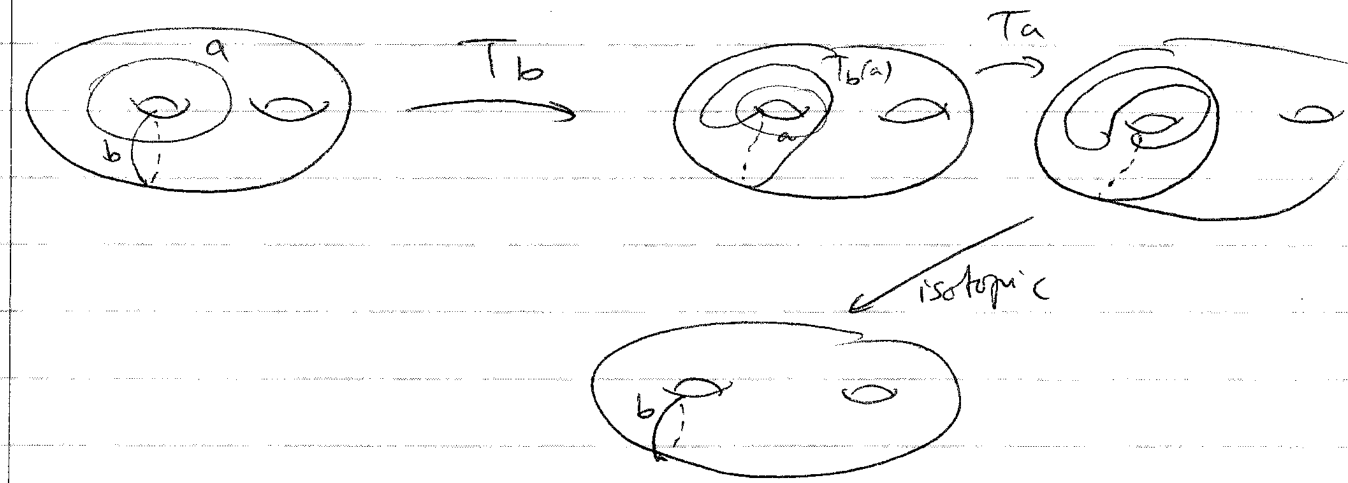
Note: $\forall \alpha$ nonseparating, $\exists h \in \mathrm{Homeo}^+(S)$ with $h(\alpha) = \beta$ (by classific of surfaces)

WLOG



$T_a T_b T_a = T_b T_a T_b$ $\rightsquigarrow$

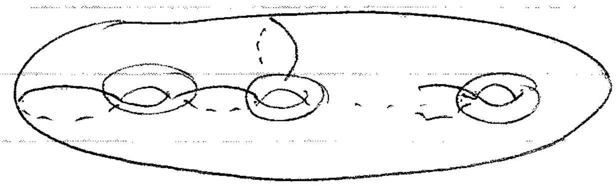
Proof of (4'): is equiv to $T_a T_b(a) = b$



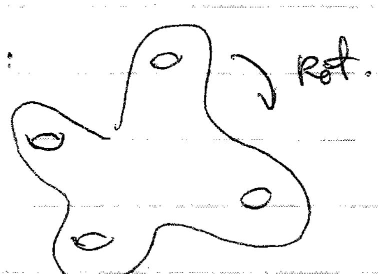
Finite generation

Theorem (Dehn 1920's) $\forall g \geq 1$, $Mod(\Sigma_g)$ is generated by finitely many twists about nonseparating simple closed curves.

Humphries: The following generates:



Exercise:



Write this as a product of twists !!

Application: Theorem (Mumford, Powell): Let $g \geq 3$.
Then $\underbrace{H_1(Mod\, g, \mathbb{Z})}_{(Mod\, g)^{ab}} = 0$

Pf: (Harer) 1) $Mod\, g = \langle \{T_\alpha \mid \alpha \text{ nonsep}\} \rangle$
2) Any 2 T_α's are conj to each other.

$\forall\, \alpha, \beta$ unsep, $\exists\, h$ s.t $h(\alpha) = \beta$

$\Rightarrow h T_\alpha h^{-1} = T_\beta$

Fix any unsep α

$\Rightarrow \Psi : \mathrm{Mod}_g \longrightarrow$ Abelian

$\Psi(\mathrm{Mod}_g) = \Psi(\langle T_\alpha \rangle)$ cyclic.

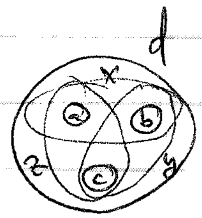
let $\Psi(T_\alpha) = t$.

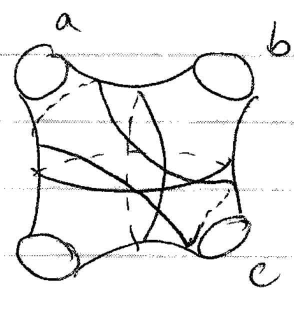
Lantern relation:

x, y, z separate the curves into pairs

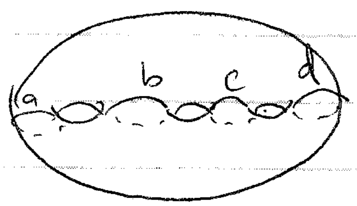
$xyz = abcd$

on $\Sigma_{0,4}$





If $g \geq 3$, $\exists$ lantern in Σ_g s.t. a, b, c, d, x, y, z are non-separating.



$$xyz = abcd$$
$$\Big\downarrow \Psi$$
$$t^3 = t^4 \Rightarrow t = 1$$

q.e.d.

Some problems related to mapping class groups

Curves on surfaces

1. (The torus)

   (a) Prove that there is a bijection from the set of homotopy classes of closed curves on T^2 and the set $\{(p, q) : p, q \in \mathbf{Z}\}$.

   (b) Prove that, under the bijection given in part (a), the simple (homotopy classes of) closed curves correspond precisely to the pairs (p, q) with p relatively prime to q.

2. Prove that a pair of simple closed curves a and b realizes $i(a, b)$ if and only if, for any points x, y in $a \cap b$, the union of the subsegments in a and b cut out by x and y do not form a "bigon", i.e. the union of two segments bounding a disk.

3. Prove that if a, b are simple closed curves in S with $i(a, b)$ odd, then both a and b are *nonseparating*, i.e. $S \setminus a$ (resp. $S \setminus b$) is still connected.

4. (Formulas for geometric intersection number)

   Let S be a surface of genus $g \geq 1$.

   (a) Prove that, for simple closed curves $a = (p, q)$ and $b = (p', q')$ in S, the geometric intersection number $i(a, b)$ is given by the formula

   $$i(a, b) = pq' - p'q$$

   (b) Let a, b be simple closed curves in S, and let N be a regular neighborhood of $a \cup b$. Prove that $i(a, b) = |\chi(N)|$; recall that the right hand side is just $|2g - 2|$, where g is the genus (=number of holes) of N.

Dehn Twists

5. (a) Prove that for any two nonseparating curves α, β in S, there exists $h \in \mathrm{Diff}(S)$ with $h(\alpha) = h(\beta)$. Conclude that Dehn twists about any two nonseparating curves are conjugate.

   (b) Extend this result to pairs of curves with intersection number one. To any collection of curves $\{\alpha_i\}$ with $i(\alpha_i, \alpha_j) = 0$ and so that $\bigcup_i \alpha_i$ doesn't separate S.

6. Write down a proof that for any $f \in \mathrm{Mod}(S)$ and Dehn twist T_α, we have the following equality in $\mathrm{Mod}(S)$:

   $$T_{f(\alpha)} = f T_\alpha f^{-1}$$

7. Prove the following result of J. McCarthy, which is a converse to the braid relation: if a, b are simple closed curves, and if T_a and T_b satisfy the braid relation $T_a T_b T_a = T_a T_b T_a$, then it must be that $i(a, b) = 1$.

8. (The chain relation)

(a) Suppose $i(\alpha,\beta)=1$. Show that a regular neighborhood of $\alpha\cup\beta$ in S is a genus one surface with boundary γ, and that γ is separating in S.

(b) Prove that $(T_\alpha T_\beta)^6 = T_\gamma$.

(c) Part (b) is a special case of a more general phenomenon. We say that an ordered set $c_1,\dots,c_n$ of simple closed curves on S forms an *n-chain* if $i(c_k,c_{k+1})=1$ for $1\le k\le n-1$ and $i(c_k,c_l)=0$ if $|k-l|\ge 2$. If n is odd, show that the boundary of a regular neighborhood of any n-chain has two components d_1 and d_2; if n is even show that the boundary has one component d. Prove the *chain relation* in Mod(S), which says that for a given n-chain $c_1,\dots,c_n$, if n is odd then

$$(T_{c_1}T_{c_2}\cdots T_{c_n})^{n+1} = T_{d_1}T_{d_2}$$

and if n is even then

$$(T_{c_1}T_{c_2}\cdots T_{c_n})^{2n+2} = T_d.$$

Birman exact sequence

9. Let S be a surface, and let $x\in S$ be a marked point, which we can also think of as a "puncture". The *mapping class group of S "punctured at x"*, denoted Mod$(S\setminus\{x\})$, is defined the be the group of homotopy classes of homeomorphisms of S, where both the homeomorphisms and the homotopies are required to fix x at all times.

Let α be a simple closed curve with basepoint x, and let $\mathrm{Push}_\alpha\in\mathrm{Homeo}^+(S)$ denote the homeomorphism which is "take your finger and push the point x along the curve α". The Birman exact sequence is the following:

$$1\to\pi_1(S)\xrightarrow{i}\mathrm{Mod}(S\setminus\{x\})\xrightarrow{\pi}\mathrm{Mod}(S)\to 1 \qquad (1)$$

where $i(a):=\mathrm{Push}_a$ and where *pi* is the homomorphism which is "forget that your homeomorphism and its homotopy class happen to fix x". Prove that the Birman exact sequence is indeed exact.

10. (a) Let c and d be the two boundary components of an annulus which is a closed tubular neighborhood of a in S. Prove that $\mathrm{Push}_a = T_cT_d^{-1}$. In particular, the kernel of (1) is generated by Dehn twists about nonseparating curves.

(b) Let S be closed, and let $x_1,\dots,x_n\in S$. Prove that Mod$(S\setminus\{x_1,\dots x_n\})$ is finitely generated by nonseparating curves if Mod(S) is finitely generated by nonseparating curves. This is crucial for the inductive step in the proof that the mapping class group of a closed surface is finitely generated.

11. (Surfaces with boundary) Let S be a surface with a single boundary component. Let $\mathrm{Mod}(S)$ be the usual mapping class group, where all homeomorphisms and homotopies fix ∂S pointwise. Let $\widehat{Mod}(S)$ denote the group of homotopy classes of homeomorphisms of S, where both homeomorphisms and homotopies are now allowed to move points in ∂S (of course ∂S itself must be invariant).

(a) Let S' be the surface obtained from S by gluing a punctured disk to S along its boundary. Prove that the natural map $\widehat{Mod}(S) \to \mathrm{Mod}(S')$ is an isomorphism.

(b) Prove that there is an exact sequence

$$1 \to \mathbf{Z} \to \mathrm{Mod}(S) \to \widehat{\mathrm{Mod}}(S) \to 1$$

where the kernel is generated by a Dehn twist about a curve α isotopic to ∂S.

(c) Let Mod_g denote the mapping class group of the closed surface Σ_g, and let $\mathrm{Mod}_{g,1}$ denote the mapping class group of Σ_g with an open disk removed, i.e. of the genus g surface with one boundary component. Let $T^1\Sigma_g$ denote the unit tangent bundle of Σ_g. Note that there is an exact sequence

$$1 \to Z \to \pi_1(T^1\Sigma_g) \to \pi_1(\Sigma_g) \to 1$$

where the kernel is generated by π_1 of the fiber of $T^1\Sigma_g$. Prove a version of the Birman exact sequence for a surface S with one boundary component by showing that the sequence

$$1 \to \pi_1(T^1\Sigma_g) \to \mathrm{Mod}_{g,1} \to \mathrm{Mod}_g \to 1$$

is exact.