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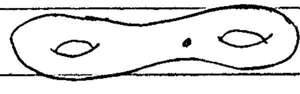
$K(G,1)$ is a CW-complex with $\pi_1 = G$, connected and its univ. covering space is contractible.



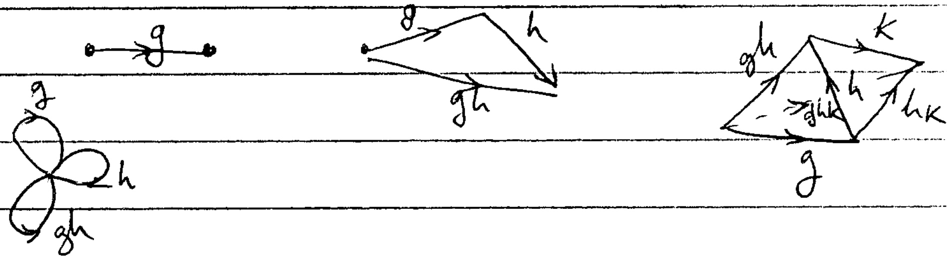
We can always take a finite 0-skeleton, a 1-skeleton if G is f.g., a 2-skeleton if it is f. presented.

G is type F if it has a finite $K(G,1)$

type F_n if $\exists\, K(G,1)$ with finite n-skeleton.

F_∞ $\exists\, K(G,1)$ with all skeleton finite, that is F_n, $\forall n \in \mathbb{N}$.

The <u>bar construction</u>: n-cells G^n



$$H^*(C_n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/n & \text{if } * \text{ even}, >0 \\ \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{if } * \text{ is odd} \end{cases}$$

In particular, any non-trivial finite group is F_∞ but not F. Type F also implies torsion free.

Take the universal cover of a $K(G,1)$. G is acting freely on it. This is a G-CW complex E, which is contractible.

The cellular chain complex $C_*(E)$ is a chain complex of free $\mathbb{Z}G$-modules, which is exact except it needs a $\mathbb{Z}$ added in degree -1.

Hence it is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$ module.

Types: FL – $\exists$ finite free resolution of $\mathbb{Z}$ by fin. gen. frees
FP – $\exists$ finite res of $\mathbb{Z}$ by f.g. projectives
FP_∞ – $\exists$ res of $\mathbb{Z}$ by f.g. projectives
FP_n – $\exists$ res of $\mathbb{Z}$ in which first n-terms are f.g. projs.
Same for FL_∞, FL_n
FH – $\exists$ a free G-CW complex which is acyclic & finitely many orbits of cells.
FH_∞
FH_n

Let M be a module over a ring R (could be $\mathbb{Z}$)

$$\begin{array}{ccccccccccc}
 & & \cdots & P_2 & \to & P_1 & \to & P_0 & \to & M & \to 0 \\
 & & & 0 & \to & \oplus & \xrightarrow{id} & \oplus & & & \\
 & & & & & Q_0 & & Q_0 & \to & 0 & \\
0 & \to & Q_1 & & \to & \oplus & & & & & \\
 & & & & & Q_1 & \to & 0 & & &
\end{array}$$

Pick Q_0 s.t. $P_0 \oplus Q_0$ ~~is~~ free

$FL_n \equiv FP_n$
$FL_\infty \equiv FP_\infty$

Difference (if any) between FP & FL is measured by $K_0(R)$ (for R-modules)
Start with the monoid of isom. classes of f.g. projectives, make an abelian group by formally adding inverses $\mathbb{N} \rightsquigarrow \mathbb{Z}$.
Example: $R = \mathbb{Z}/6$ $\quad \mathbb{Z}/_2$ & $\mathbb{Z}/_3$ are projective but not free. The monoid of projectives is $\mathbb{N}^2$ the group $K_0(R)$ is $\mathbb{Z}^2$.

If M is FP, with FP resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

then M is FL iff $\sum (-1)^i [P_i] = [R^{\oplus k}]$ in $K_0(R)$ for some k.

Group homology / cohomology detects FP.

TFAE: (1) A module M is FP_n over R

(2) $\forall$ exact limit (for ex. a direct product) N_*, the map $\mathrm{Tor}_k^R(\lim N_\alpha, M) \to \lim \mathrm{Tor}_k^R(N_\alpha, M)$ is iso for $k<n$ and epi for $k=n$

(3) For any indexing set I, the map $\mathrm{Tor}_k^R(\prod_I R, M) \to \prod_I \mathrm{Tor}_k^R(R, M)$ is iso for $k<n$ & epi for $k=n$.

~~(4) For any exact co~~

(1)$\Rightarrow$(2) easy

(2)$\Rightarrow$(3) trivial, just a special case.

(3)$\Rightarrow$(1) ~~take $I=M$~~ by induction.

$$K \rightarrowtail F \twoheadrightarrow M$$

K is FP_{n-1} $\Leftrightarrow$ M is FP_n

Case $n=0$ take $I=M$. Then

$$\left(\prod_{m\in M} R\right) \otimes M \twoheadrightarrow \prod_{m\in M} M$$

Have an element of RHS $\underline{m}$ in M_m

$$\sum \underline{r}^{(i)} \otimes m_i \longrightarrow \underline{m}$$

these finitely many m_i's generate.