I Definitions & Examples

Coxeter graph

Γ = finite, simplicial, labelled graph

vertices : $S = \{s_1, \dots, s_n\}$

edges : $s_i \overset{m_{ij}}{\text{———}} s_j$ $\quad m_{ij} \geq 2$

(convention : $m_{ij} = \infty$ if no edge)

Coxeter group

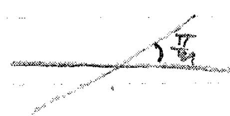
$$W_\Gamma = \langle S \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \rangle$$

= discrete linear gp generated by reflections

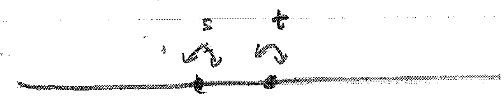
reflection (w.r.t $v \in \mathbb{R}^n$, $\|v\|_B = 1$):

$r_v(w) = w - 2B(v,w)v$ $\qquad B$ = bilinear form

Eg 1) $n=2$: $W = \langle s, t \mid s^2 = t^2 = 1, (st)^m = 1 \rangle = D_{2m}$



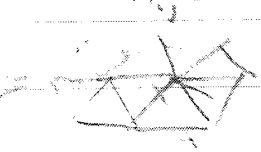
if $m = \infty$: $W = \mathbb{Z}_2 * \mathbb{Z}_2 = D_\infty$



2) $n=3$: $W = T(\underbrace{p,q,r}_{m_{ij}\text{'s}}) =$

reflect across walls of triangle

$T(3,3,3) =$



3) $n \geq 5$: $W = \langle s_1, \dots, s_n \mid \underbrace{s_i s_{i+1} = s_{i+1} s_i}_{\text{adjacent walls}} \rangle$

"right-angled" Coxeter groups

$\mathbb{H}^2 =$

4) $n > 2$ $\quad W = \Sigma_n$ = symm. gp on n letters $= \langle s_1, \dots, s_{n-1} \mid$ $\qquad s_i = (i\ i{+}1)$

Remark: can rewrite $(s_i s_j)^{m_{ij}} = 1$ as $s_i s_j s_i \cdots = s_j s_i s_j \cdots$

Artin group

$$A_\Gamma = \langle S \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \rangle$$

$$= \pi_1 \begin{pmatrix} \text{hyperplane complement} \\ \text{associated to } W_\Gamma \end{pmatrix}$$

$W \curvearrowright \mathbb{R}^n$ as reflection gp $\rightsquigarrow W \curvearrowright \mathbb{C}^n$

$Y =$ non-singular pts of this action

$$= \mathbb{C}^n - \bigcup_{\substack{r \in W \\ \text{reflection}}} H_r \quad \leftarrow \text{fixed by } r$$

$Y \longrightarrow Y/W$ covering space

Thm: (Brieskorn '71) $\pi_1(Y/W) = A$

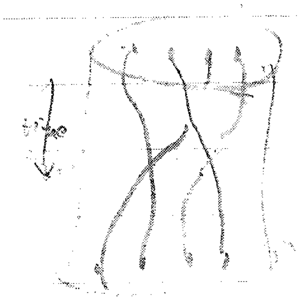
Eg: $W = \Sigma_n =$ symm gp on n letters $\curvearrowright \mathbb{R}^n \rightsquigarrow W \curvearrowright \mathbb{C}^n$

reflection hyperplanes: $H_{ij} = \{(z_1, \ldots, z_n) \mid z_i = z_j\}$

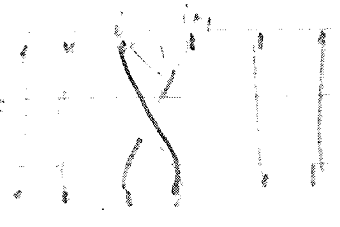
$Y/W = \mathbb{C}^n - \bigcup H_{ij} / W =$ config space of n distinct (unordered) pts in $\mathbb{C}$

$\pi_1(Y/W) =$ braid gp on n strands

$= \langle s_1, \ldots, s_n \mid s_i s_j = s_j s_i, \ \ldots \rangle$



$s_i =$



Remark: All Artin gps are infinite, in fact they are conjectured to be torsion-free.

Eg: $\Gamma = \underset{s}{\bullet}\overset{4}{\text{———}}\underset{t}{\bullet}$ $\quad \overset{W = D_8}{A} = \langle s, t \mid \underbrace{stst}_{\Delta} = tsts \rangle$ is big!

Δ is central: $t(\underbrace{stst}_{\Delta}) = (\underbrace{tsts}_{\Delta})t$, Center$(A) = \langle \Delta \rangle$

$A / \langle\langle \Delta \rangle\rangle = \langle \overset{st}{\overset{\shortparallel}{x}}, \overset{s}{\overset{\shortparallel}{y}} \mid x^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}$ $\qquad tsts = s^{-1}(\overset{\shortparallel}{stst})s = y^{-1} x^2 y$

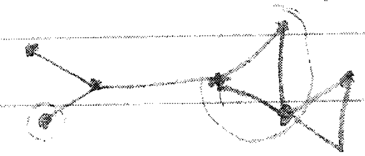
Def: A_Γ is finite type (or spherical) if W_Γ is finite

Notation: $T \subseteq S$, W_T, A_T

II. Geometry: complexes associated to W_Γ, A_Γ

For $T \subseteq S$,

W_T = subgp of W gen by T = Coxeter gp assoc to subgraph of Γ spanned by T

Γ = 

A_T = subgp of A gen by T =

Davis Complex for W (Tits, Davis)

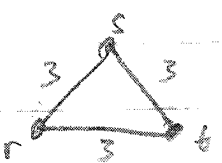
$$\mathcal{D}_W = |\{wW_T \mid T \subseteq S,\ W_T \text{ finite}\}|$$

stab$(wW_T) = wW_Tw^{-1}$ finite!

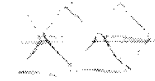
$W \curvearrowright \mathcal{D}_W$ left mult, proper, cocompact

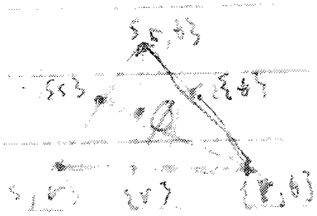
$$K = |\{W_T \mid T \subseteq S,\ W_T \text{ finite}\}| \quad \text{fund domain}$$

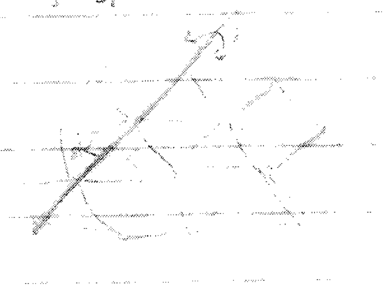
$$\mathcal{D}_W = W \times K/\sim$$

Eg: Γ = 

W_Γ = Euclidean triangle gp



K: 

$\mathcal{D}_W$ = 

Remark: generators s_i act as "reflections" on $\mathcal{D}_W$, fixed set of reflection called "wall"

What's $\mathcal{D}_W$ good for?

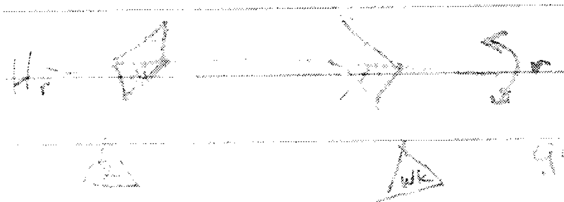
1) Dictionary:

combinatorial prop of W $\longleftrightarrow$ geometric prop of $\mathcal{D}_W$

$w = s_{i_1} \cdots s_{i_k}$ word — "gallery" in $\mathcal{D}_W$ from K to wK

minimal length word — gallery crosses each wall at most once

non-minimal $\Rightarrow$ $w = s_{i_1} \cdots \boxtimes \cdots \boxtimes \cdots s_{i_k}$ "Exchange Condition" — non-minimal $\Rightarrow$ can shorten by reflecting



3) nice geometry:

Davis: $\mathcal{D}_W$ contractible

Moussong: $\mathcal{D}_W$ CAT(0) $\Rightarrow$ W is CAT(0) gp

$\mathcal{D}_W$ hyperbolic $\Leftrightarrow$ W has no $\mathbb{Z}^2$ subgps $\Updownarrow$ W is word hyperbolic

3) Interesting spaces in their own right.

Deligne complex for A (Deligne, C-Davis, v.d.Lek)

$$\mathcal{D}_A = \left| \{ a A_T \mid T \subseteq S,\ A_T \text{ finite type} \} \right| \qquad A_T \text{ finite type} \overset{\text{def}}{\iff} W_T \text{ finite}$$

$$= A \times K / \sim$$

$A \curvearrowright \mathcal{D}_A$, stabilizers $a A_T a^{-1}$ are not finite
cocompact but not proper!
$\mathcal{D}_A$ not locally finite!

Salvetti complex for A (Salvetti)

$A \underset{\sigma}{\twoheadrightarrow} W$ has a set theoretic section:
$w \in W$, write a shortest word $w = s_{i_1} s_{i_2} \cdots s_{i_k}$
set $\sigma(w) = s_{i_1} s_{i_2} \cdots s_{i_k} \in A$

Write $\widehat{W} = \sigma(W)$, $\widehat{W}_T = \sigma(W_T)$

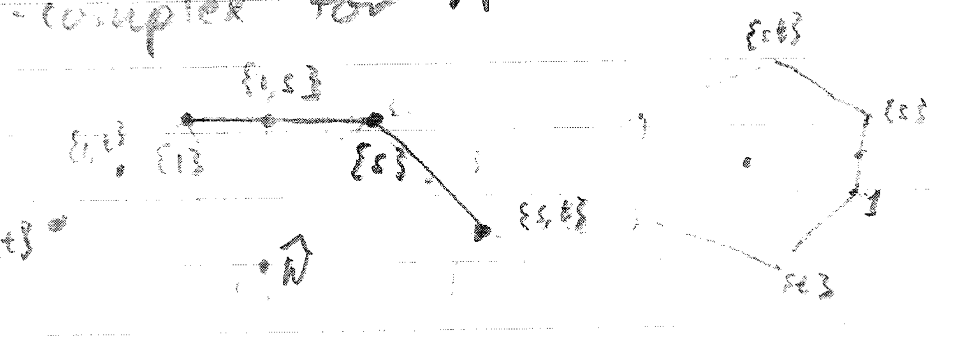
$$\mathcal{S}_A = \left| \{ \underset{\text{set}}{a \widehat{W}_T} \mid T \subseteq S,\ W_T \text{ finite} \} \right|$$

$A \curvearrowright \mathcal{S}_A$ cocompact, free!

Eg: Γ: $\underset{s}{\bullet} \overset{m}{\text{———}} \underset{t}{\bullet}$ $\quad W = D_{2m}$

exercise: $\mathcal{S}_A \underset{\wedge}{=}$ bary. subdiv Cayley 2-complex for A

$a \widehat{W}_\emptyset = \{a\}$



Topology of $\mathcal{D}_A$, $\mathcal{S}_A$

Fact: $\underset{\text{Deligne}}{\mathcal{D}_A \simeq} \widetilde{Y} \underset{\text{Salvetti}}{\simeq} \mathcal{S}_A$ $\qquad \widetilde{Y}$ = univ cover of hyp comp

$\widetilde{Y} \to Y \xrightarrow{\text{cov}} Y/W$

Conj. $\widetilde{Y}$ is contractible

Conj $\Leftrightarrow$ $\widetilde{Y}/A \overset{Y/W}{=}$ is a $K(A,1)$-space

$\Leftrightarrow$ $\mathcal{S}_A/A$ is a (finite) $K(A,1)$-space

$\Rightarrow$ A type F, torsion-free, ...

- (Deligne) conj true for finite type A.
- (C-Davis) two metrics on $\mathcal{D}_A$
  - Moussong metric : CAT(0) if $\mathcal{D}_A$ 2-dim'l (some 3-dim)
  - cubical metric : CAT(0) $\Leftrightarrow$ A is "FC-type"
- (C+Crisp) : suff cond for CAT(-1) metric $\Rightarrow$ A (weakly) rel hyp

Techniques + Open Questions

① A finite type ($\Leftrightarrow$ W finite)

Powerful combinatorial techniques : Garside structure
(Garside, Deligne, Birman-Ko-Lee, Brady, Bessis)

A^+ = monoid of positive words

partial order on A^+ : $a \leq b$ if $\exists c \in A^+$, $ac = b$.

Key facts

- A^+ is a lattice wrt $\leq$ ($\exists$ l.u.b's and g.l.b's)
- $\exists \Delta \in A^+$ s.t. $A = A^+[\Delta^{-1}]$

Let $M = \{a \in A^+ \mid a \leq \Delta\} \supset S$

Garside structure $\rightsquigarrow$ canonical forms for A as words in M

$a \in A^+$: $m_1 = g.l.b(a, \Delta) \Rightarrow a = m_1 a_1$

$m_2 = g.l.b(a_1, \Delta) \Rightarrow a = m_1 m_2 a_2$

$a = m_1 m_2 \cdots m_k$ canonical form

$g \in A^+ \Rightarrow g = a\Delta^{-n}$, $a \in A^+$, n minimal

$\Rightarrow g = m_1 \cdots m_k \Delta^{-n}$ canonical form.

Garside structure

$\Rightarrow$ canonical forms for elts. of A

$\Rightarrow$ biautomatic structure on A

$\overset{\text{Deligne}}{\Rightarrow}$ Σ contractible

$\Rightarrow$ S_A/A finite $K(\pi,1)$.

Q. Is A a CAT(0) group?

(2) A ~~affine~~ Euclidean type

For $\tilde{A}_n, \tilde{C}_n, \tilde{B}_n$, conj true.
For $\tilde{A}_n, \tilde{C}_n$ $\quad A \underset{\text{finite index}}{<} \mathrm{Mod}$(punctured sphere)

Q: (McCammond) Is there an analogue of a Garside structure ~~for~~ affine type A?

(3) Large type ($\subseteq$ 2-dim'l)
all $m_{ij} \geq 3 \implies \mathcal{D}_A$ 2-dim'l

combinatorial techniques ~ small techniques
Appel-Schupp, Peifer, Juhász
$m_{ij} \geq 4 \implies$ biautomatic

geometric techniques: $\mathcal{D}_A$ CAT(0) $\Rightarrow$ conj true

Q: Does A act properly on a CAT(0) cubical complex.

(4) FC-type
A_T finite type $\Leftrightarrow s_i, s_j \in T \Rightarrow m_{ij} < \infty$ (no Euclidean: any hyp sub-groups)

combinatorial normal forms ~~ms~~ automatic structure
geometric $\mathcal{D}_A$ CAT(0) $\Rightarrow$ conj true
Q: Conj. $\mathcal{D}_A$ ~~is~~ hyperbolic $\Rightarrow$ $\mathcal{D}_A$ hyperbolic

(5) RAAG's
A is right angled if $m_{ij} = 2, \infty$ $\forall i,j$