$\boxed{\text{Benson Farb}}$ II

Analogy

Vector space	Surface
$\mathbb{H}^n$	Teich (S)
lattice Γ	Mod (S)
$\Gamma \backslash \mathbb{H}^n$ complete finite vol hyp orbifold	$\mathcal{M}(S)$ = moduli space
$\varphi \in \Gamma$ { finite order, parabolic, hyperbolic	$\varphi \in$ Mod S { finite order, reducible, pseudo-Anosov
unique axis in $\mathbb{H}^n$ for hyp φ	unique axis in Teich (S) for φ pseudo
Jordan form	Thurston normal form
eigenspaces, foliations by parallel copies	invariant subsurfaces, invar foliations
eigen-values	dilatations on pseudo or subsurf
$\partial \mathbb{H}^n$	$\mathcal{PMF}$, Thurston bdry

Goals: Σ_g = closed genus g surf.
$\mathrm{Mod}_g = \mathrm{Mod}(\Sigma_g)$

(1) What does an arbitrary elem $\varphi \in \mathrm{Mod}_g$ look like? $\rightarrow$ think eigen-values, Jordan form of a matrix

(2) Describe $\mathcal{M}_g$ = moduli space
= {hyp metrics on Σ_g}/~
= {conformal classes of Riem. metrics}
= {complex structures}
= {alg. curves}
⋮

The torus

Recall $\mathrm{Mod}(T^2) \cong SL_2\mathbb{Z}$

$$\varphi_A \longleftrightarrow A$$

Prop: Every $A \in SL_2\mathbb{Z}$ is conj to one of:

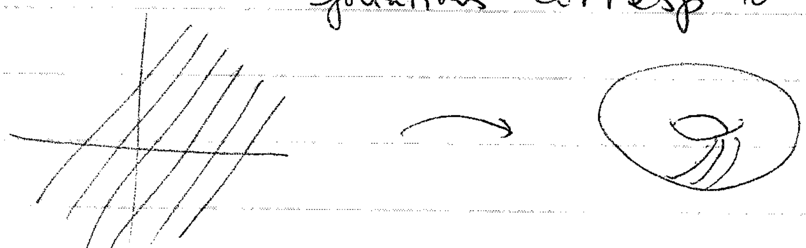
(1) finite order

(2) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ some $n \in \mathbb{Z}$

(3) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, $\lambda > 1$ $\rightsquigarrow$ case when $|\mathrm{Tr} A| > 2$

Structure of case (3)

1. There are 2 invariant irrational foliations corresp to eigenspaces of A.



2. area-preserving
3. periodic pts for φ_A are dense in the torus
4. $\forall$ scc $\beta, \gamma \subset T^2$, $i(\varphi_A^n(\gamma), \beta) \simeq \lambda^n$.
5. entropy.

Proofs of Prop $\rightarrow$ proof #1 Jordan form

$\rightarrow$ proofs #2, 3 Look at action on $\mathbb{H}^2$

#2 Look at minset(A)

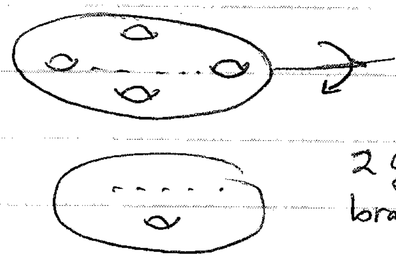
#3 A acts on $(\mathbb{H}^2 \cup \partial\mathbb{H}^2) \cong \overline{D}^2$. Apply Brouwer's fixed pt.

An example in higher genus.

Σ_g

2 fold branch cover $\downarrow$

T^2



$2g-2$ branch pts

Let A be a hyp matrix ($|\mathrm{Tr} A| > 2$)

$\varphi_A \in \mathrm{Homeo}^+(T^2)$

Check that φ_A^m lifts for some m to a homeo φ of

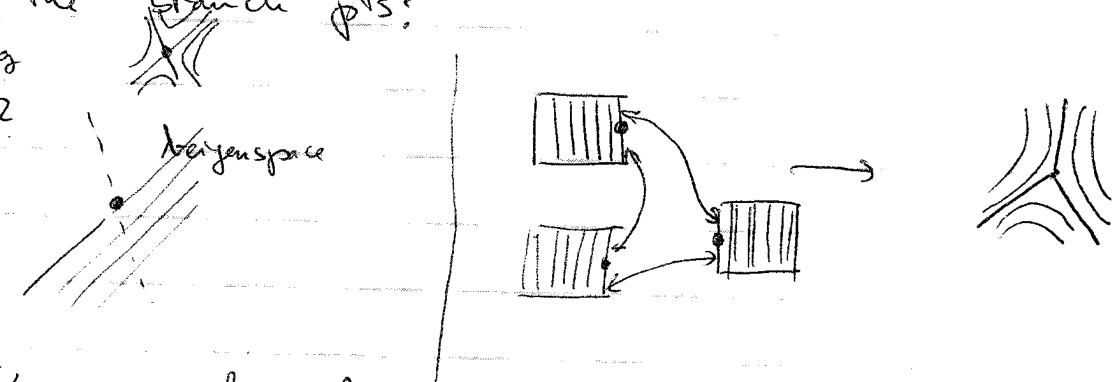
What does Ψ look like?

$\exists$ piecewise flat metric on Σ_g s.t. Ψ is at least locally affine. If $z = x+iy$ $\varphi(z) = \lambda x + i\lambda^{-1} y$.

Near the branch pts:

Σ_g

T^2



Ψ is pseudo-Anosov.

Ψ has great properties (see 1-5 above)

Nielsen - Thurston classification

$g \geq 1$

Every $\varphi \in \mathrm{Homeo}^+(\Sigma_g)$ is homotopic to some $\Psi \in \mathrm{Homeo}^+(\Sigma_g)$ with:

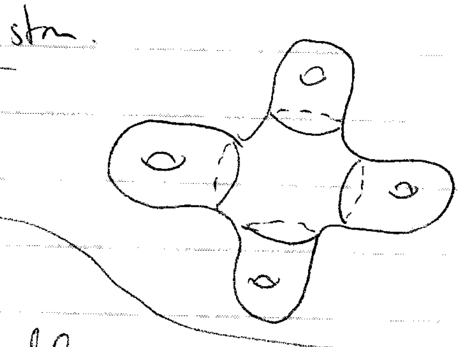
(1) Ψ is finite order

(2) Ψ is reducible, i.e. $\exists$ scc $\gamma \subset \Sigma_g$, $\exists\, m \geq 1$ with $\Psi^m(\gamma) \sim \gamma$

(3) Ψ is pseudo-Anosov

Cor: Norm. form Thurston.

$\varphi \in \mathrm{Mod}_g$ is



Applications:

- classif of surface bundles over a circle $\Sigma_g \to M^3 \downarrow S^1$
- group theory of mods

Teich Space Basics

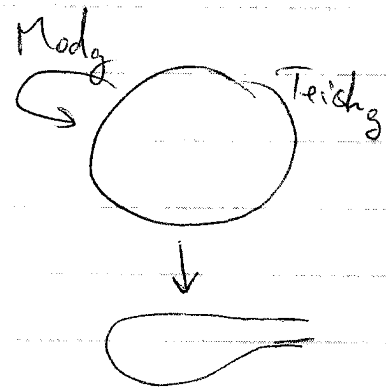
$$\mathrm{Teich}_g = \{(X,f) \mid X \text{ hyperb surface}, \ f:\Sigma_g \xrightarrow{\approx} X\}/\sim \qquad g \geq 1$$

$$\mathrm{Teich}_1 = \mathbb{H}^2$$

$$\mathbb{R}^2/\mathrm{Span}\{\tbinom{1}{0},\tbinom{0}{1}\} \qquad \mathbb{R}^2/\mathrm{Span}\{\tbinom{1}{0},\tbinom{1}{1}\}$$

$\downarrow$ different in Teich_1, even though they are isom. tori.

- Mod_g acts on Teich_g via $\varphi.(X,f) := (X, f\circ\varphi^{-1})$

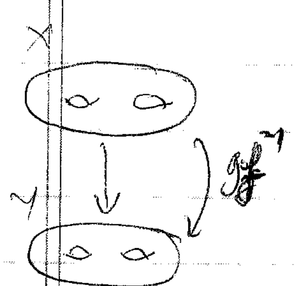


$$\mathcal{M}_g := \mathrm{Mod}_g \backslash \mathrm{Teich}_g$$

$$\pi_1^{orb}(\mathcal{M}_g) = \mathrm{Mod}_g.$$

Define $d_{\mathrm{Teich}_g}((X,f),(Y,g)) :=$

$$= \frac{1}{2} \inf_{h \sim g\circ f^{-1}} \log K(h)$$



where $K(h) = \sup_{x\in X} K_x(h)$ quasiconf dil of h at x.

this is a metric, complete

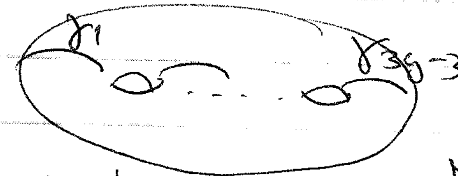
Mod_g acts by isometries.

Theorem (Fricke):

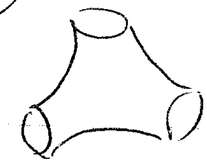
$$\mathrm{Teich}_g \simeq \mathbb{R}^{6g-6}.$$

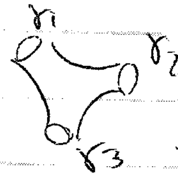
"Proof": $\mathrm{Teich}_g \to (\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$

$$X \mapsto (\ell_X(\gamma_1), \dots, \ell_X(\gamma_{3g-3}), \theta_1(X), \dots, \theta_{3g-3}(X))$$



pants decomp.



Key: $\mathrm{Teich}\Big($  $\Big) \xrightarrow{\simeq} (\mathbb{R}^+)^3$

$$X \longmapsto (\ell_X(\gamma_1), \ell_X(\gamma_2), \ell_X(\gamma_3))$$

Theorem (Fricke) Action of Mod_g on Teich_g is properly discontinuous.

Cor: $H^*(\mathcal{M}_g, \mathbb{Q}) \simeq H^*(\mathrm{Mod}_g, \mathbb{Q})$.

Proof of the Thurston classif. - ideas

Given $\varphi \in \mathrm{Mod}_g$, φ acts by isom of $(\mathrm{Teich}_g, d_{\mathrm{Teich}})$

Look at $\tau(\varphi) = \inf_{x \in \mathrm{Teich}_g} d(\varphi(x), x)$

3 cases
- $\to \tau(\varphi) = 0$ inf realized.
- $\to \tau(\varphi) > 0$ and realized
- $\to \tau(\varphi)$ not realized.