Lecture 3:
Asymptotic cones

Slides avaiable at:

http://www.math.yale.edu/∼bk255/talk1.pdf
http://www.math.yale.edu/∼bk255/talk2.pdf
http://www.math.yale.edu/∼bk255/talk3.pdf

References:

Metric spaces of nonpositive curvature, Bridson-Haefliger.

Manifolds of nonpositive curvature, Ballmann-Gromov-Schroeder.

Boundaries of hyperbolic groups, Benakli-Kapovich.

Rigidity for quasi-isometries of symmetric spaces and Euclidean buildings, Kleiner-Leeb.

The asymptotic geometry of negatively curved spaces: uniformization, geometrization, and rigidity, 2006 ICM.

Thm. (Morrey-Bers) If ξ is a bounded measurable conformal structure on S^2, then there is a quasi-Mobius homeomorphism $f : S^2 \to S^2$ such that $f_*\xi$ is the standard conformal structure ξ_0.

Thm. (Sullivan) Any action $G \curvearrowright S^2$ of a countable group by uniformly quasi-Mobius homeomorphisms is quasi-Mobius conjugate to an action by Mobius transformations.

Thm. (Gromov, Tukia) If $n \geq 3$, then any uniformly quasi-Mobius action $G \curvearrowright S^n$ which is cocompact on triples is quasi-Mobius conjugate to a Mobius action.

Cor. (Gromov, Cannon-Cooper) If $n \geq 3$, then any group quasi-isometric to $\mathbb{H}^n$ admits a discrete, cocompact, isometric action on $\mathbb{H}^n$.

The theory of quasi-Mobius homeomorphisms for Carnot-Caratheodory spaces was developed by Mostow, Pansu, and others. It permits one to implement these arguments for other rank 1 symmetric spaces.

For more general boundaries, the situation has been developing rapidly in the last 10 years, but the theory is still primitive compared to the rank 1 case.

Boundaries of hyperbolic groups

Let $G \curvearrowright X$ be a discrete, cocompact, isometric action on a proper Gromov hyperbolic space, and equip ∂X with a visual metric.

• ∂X is **Ahlfors Q-regular** for some $Q \in [0, \infty)$. This means that the Q-dimensional Hausdorff measure of every r-ball is comparable to r^Q, when $r \in [0, \mathrm{diam}(\partial X)]$.

• ∂X is uniformly perfect, unless G is virtually cyclic. This means that for some $\rho \in [1, \infty)$, every metric annulus

$$B(p, \rho\, r) \setminus B(p, \frac{r}{\rho})$$

is nonempty, when $r \in (0, \mathrm{diam}(\partial X)]$.

• If G is one-ended, then ∂X is **linearly connected**, i.e. there is a constant $D \in [1, \infty)$ such that for every $x, y \in \partial X$, there is a path of diameter at most $D\, d(x, y)$ between x and y.

• If G does not virtually split of a virtually cyclic group, then ∂X is **LLC**. This means that there is a $\lambda \in (0, \infty)$ such that for all $r \in (0, \text{diam}(\partial X)]$ the inclusions

$$B(x, \lambda r) \to B(x, r), \quad \partial X \backslash B(x, r) \to \partial X \backslash B(x, \lambda r)$$

induce zero on reduced 0-dimensional homology.

Recent work on the analytic side starting with Heinonen-Koskela and Cheeger has opened the possibility of extending rank 1 rigidity arguments to a much larger class of hyperbolic groups.

In the terminology of Heinonen-Koskela, one needs to know that ∂G is quasi-Mobius homeomorphic to an Ahlfors Q-regular, Q-Loewner metric space Z.

Finding such a metric may be viewed as a kind of geometrization problem. For boundaries of groups, such metrics are characterized as Ahlfors regular metric spaces of minimal Hausdorff dimension.

Rigidity in the presence of flats

The Morse Lemma fails when there are subspaces quasi-isometric to $\mathbb{R}^2$. In general, quasi-isometries between $CAT(0)$ spaces do not induce boundary homeomorphisms.

Asymptotic cones provide a powerful tool for rigidity theorems.

Idea. Given an (L, A)-quasi-isometry

$$f : X \longrightarrow Y,$$

one may take a sequence of scale factors $\lambda_k \to 0$, and view f as a map between the rescaled spaces:

$$f_k : \lambda_k X \longrightarrow \lambda_k Y.$$

Then f_k is an $(L, \lambda_k A)$-quasi-isometry. A limit of this sequence should be an $(L, 0)$-quasi-isometry, i.e. an L-biLipschitz homeomorphism.

One can often apply topological arguments to the limit, to deduce something about the original quasi-isometry f.

Some applications of asymptotic cones

• Gromov's Theorem on groups of polynomial growth. Wilkie-Van Den Dries.

• Degeneration of hyperbolic structures.

• Quasi-isometric rigidity of various spaces: Haken 3-manifolds, symmetric spaces and Euclidean buildings, products.

• Relative hyperbolicity.

One typically proves a geometric bounds by a compactness-type argument.

The nonexistence of a uniform bound implies the existence of a sequence of configurations where the failure is worse and worse. By rescaling and passing to a limit, one arrives at an impossible configuration.

Nonprincipal ultrafilters

Def. A **nonprincipal ultrafilter** is a finitely additive measure ω on $\mathbb{N}$ such that $\omega(S) \in \{0, 1\}$ for every subset $S \subset \mathbb{N}$, and every finite subset has ω measure zero.

The existence of nonprincipal ultrafilters follows from the Axiom of Choice.

Obs. Let ω be a nonprincipal ultrafilter.

- If

$$\mathbb{N} = S_1 \cup \ldots \cup S_k,$$

then one of the S_i's has full ω-measure.

- If S, $S' \subset \mathbb{N}$ are disjoint, then at least one of them has measure zero.

- If

$$\mathbb{N} = S_1 \sqcup \ldots \sqcup S_k$$

then precisely one of the S_i's has full measure, and the others have zero measure.

Lemma. If K is a compact metric space and $\{x_k\} \subset K$, then there is a unique point $x_\omega \in K$ such that

$$\{k \in \mathbb{N} \mid x_k \in B(x_\omega, r)\}$$

has full ω-measure for all $r \in (0, \infty)$.

Def. The point x_ω is called the ω-limit of $\{x_k\}$.

Thus every bounded sequence in $\mathbb{R}$ has a well-defined ω-limit. Any any sequence of nonnegative real numbers has an ω-limit in $[0, \infty) \cup \{\infty\}$.

Ultralimits of metric spaces

Generalized metrics. If S is a set, a **generalized metric on** S is a function

$$d : S \times S \to [0, \infty) \cup \{\infty\}$$

which is symmetric and satisfies the triangle inequality.

The function d induces two equivalence relations:

$$\{d = 0\}, \quad \{d < \infty\}.$$

As usual, we may collapse the cosets of the $\{d = 0\}$ relation. If we restrict d to a coset of the relation $\{d < \infty\}$, we then obtain a metric space in the usual sense.

Let $\{(X_k, d_k)\}$ be a sequence of metric spaces.

We put a generalized metric on the product

$$\prod_k X_k$$

using the formula

$$d_\omega((p_k), (q_k)) := \omega - \lim\ d_k(p_k, q_k).$$

To obtain a metric space, we choose a basepoint $(\star_k) \in \prod_k X_k$, consider the set of points at finite distance from $(\star_k)$, and collapse the zero diameter subsets. This is the ω-**limit** of the sequence $\{(X_k, \star_k)\}$. We will use the notation $\omega - \lim(X_k, \star_k)$ or $(X_\omega, \star_\omega)$.

An **asymptotic cone** of a sequence of metric spaces $\{X_k\}$ is an ultralimit of a sequence of the form

$$\{(\lambda_k X_k, \star_k)\}$$

where $\star_k \in X_k$ and $\omega - \lim \lambda_k = 0$.

Properties of ultralimits and asymptotic cones

• A sequence $\{f_k : X_k \to Y_k\}$ of (L, A)-quasi-isometric embeddings (quasi-isometries) induces an (L, A)-quasi-isometric embedding

$$f_\omega : (X_\omega, \star_\omega) \to (Y_\omega, f_\omega(\star_\omega)).$$

• Ultralimits are always complete.

• Ultralimits of δ-hyperbolic spaces are δ-hyperbolic.

• Ultralimits of $CAT(\kappa)$ spaces are $CAT(\kappa)$.

• $[ultralimit, product] = 0$.

• $\omega - \lim(\lambda_k \mathbb{R}^n, 0)$ is isometric to $\mathbb{R}^n$.

• If X is δ-hyperbolic, then $\omega{-}\lim(\lambda_k X, \star_k)$ is an $\mathbb{R}$-tree.

Def. An $\mathbb{R}$-tree is a 0-hyperbolic metric space, or equivalently, a $CAT(-\infty)$ metric space.

• If $f_k : X_k \to Y_k$ is a sequence of (L, A)-quasi-isometries, and $\lambda_k \to 0$, then

$$f_\omega : (X_\omega, \star_\omega) \to (Y_\omega, f_\omega(\star_\omega))$$

is an $(L, 0)$-quasi-isometry.

Theorem. *Every* (L, A)*-quasi-isometry*

$$\mathbb{H}^m \times \mathbb{H}^n \to \mathbb{H}^m \times \mathbb{H}^n$$

is at distance at most $D = D(L, A)$ *from a product of quasi-isometries.*

Def. An $\mathbb{R}$-tree T **branches everywhere** if $T \setminus \{x\}$ has at least 3 connected components for every $x \in T$.

Thm. Suppose T_1 and T_2 are complete $\mathbb{R}$-trees which branch everywhere. Then any homeomorphism

$$T_1 \times T_2 \longrightarrow T_1 \times T_2$$

is a product of homeomorphisms.

For each pair of points $x, y \in \mathbb{H}^m \times \mathbb{H}^n$, let

$$\theta(x,y) \in [0, \frac{\pi}{2}]$$

be the angle to the first factor, i.e.

$$\theta(x,y) := \arctan\left(\frac{d(\pi_2(x), \pi_2(y))}{d(\pi_1(x), \pi_1(y))}\right).$$

Let $X := \mathbb{H}^m \times \mathbb{H}^n$.

Step 1. For every $\epsilon > 0$ there is an $R \in (0, \infty)$ such that if $x, y \in X$ is a horizontal pair separated by distance at least R, then

$$\theta(f(x), f(y)) \in [0, \epsilon) \cup (\frac{\pi}{2} - \epsilon].$$

Step 2. For every $\epsilon > 0$ there is an $R \in (0, \infty)$ such that either

A. For every horizontal pair $x, y \in X$ separated by distance at least R,

$$\theta(f(x), f(y)) \in [0, \epsilon).$$

B. For every horizontal pair $x, y \in X$ separated by distance at least R,

$$\theta(f(x), f(y)) \in (\frac{\pi}{2} - \epsilon].$$

Step 3. WLOG suppose we're in Case A. Then the composition f_p given by

$$\mathbb{H}^m \to \mathbb{H}^m \times \{p\} \xrightarrow{f} X$$

$$\xrightarrow{\pi_1} \mathbb{H}^m$$

is an (L', A') quasi-isometric embedding.

For every $p, p' \in \mathbb{H}^n$

$$d(f_p, f_p') < \infty.$$

Step 4. For every $x \in \mathbb{H}^m$, the set

$$\{f_p(x)\} \subset \mathbb{H}^m$$

has uniformly bounded diameter.

Lemma. Let

$$G \overset{\rho_k}{\curvearrowright} \mathbb{H}^n$$

be a sequence of isometric actions of finitely generated group on hyperbolic n-space. Then either

A. Modulo conjugation by isometries and passing to a subsequence, the actions converge (elementwise) to a limiting isometric action

$$G \overset{\rho_\infty}{\curvearrowright} \mathbb{H}^n.$$

B. G admits a fixed point free isometric action on an $\mathbb{R}$-tree.

Proof. Let $\Sigma \subset G$ be a finite generating set.

Define $\delta_k : \mathbb{H}^n \to [0, \infty)$ by

$$\delta_k(p) := \max_{g \in \Sigma} d(\rho_k(g)(p), p),$$

and

$$\Delta_k := \sup_{p \in \mathbb{H}^n} \delta_k.$$

Choose $\star_k \in X_k$ such that

$$\omega - \lim \delta_k(\star_k) < 2\Delta_k.$$

If $\omega - \lim \Delta_k < \infty$, we may pass to a subsequence so that Δ_k is uniformly bounded. Modulo conjugation, this has a convergent subsequence by Arzela-Ascoli.

If $\omega - \lim \Delta_k = \infty$, then and let

$$(X_\omega, \star_\omega) := \omega - \lim \left(\frac{1}{\Delta_k} X_k, \star_k \right).$$

The sequence of actions $\{\rho_k\}$ induces a fixed point free isometric action on the asymptotic cone.