Dimensions of Torelli groups

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MSRI

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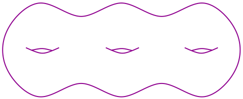
Torelli groups

S_g = surface of genus g



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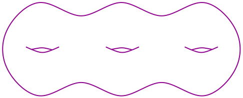

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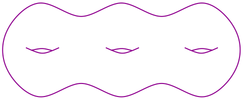

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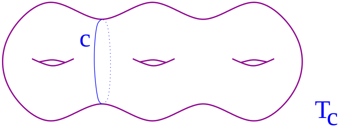
Definition of the Torelli group $\mathcal{I}(S_g)$:

$$1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1$$

Elements of the Torelli group

Dehn twists about separating curves



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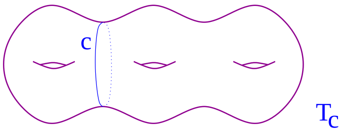
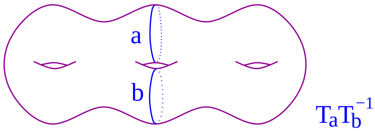


Bounding pair maps



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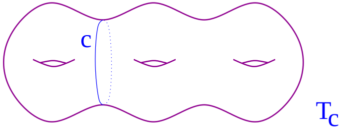

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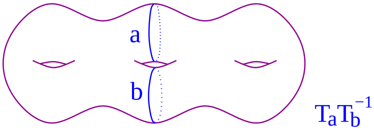

Theorem (Birman '71 + Powell '78)

These elements generate $\mathcal{I}(S_g)$.

Finiteness properties

Finite generation

Finite presentability

Finite generation of homology

Cohomological dimension

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Harer 1986, Culler–Vogtmann 1986 + Mess 1990:

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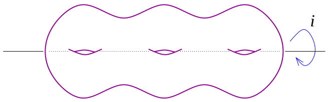

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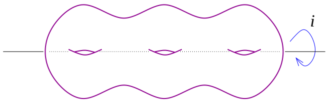

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Conjecture

For $g, k \geq 3$, $\mathrm{cd}(\mathcal{N}_k(S_g))$ is constant.

(Guess: $g - 1$)

Analogy with $\mathrm{Out}(F_n)$

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Theorem (BBM '06)

For $n \geq 3$, we have $\mathrm{cd}(\mathcal{I}(F_n)) = 2n - 4$.

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The proofs of the analogous theorems are incongruous.

Proofs

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1. $\mathrm{cd}(G) \leq D$ (Quillen)

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Then

1. $\mathrm{cd}(G) \leq D$ (Quillen)
2. $\oplus H_D(\mathrm{Stab}_G(v)) \hookrightarrow H_D(G)$
   where the sum is over a set of reps of vertices of X/G.

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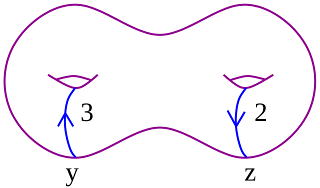
Example: $x = 3y + 2z$



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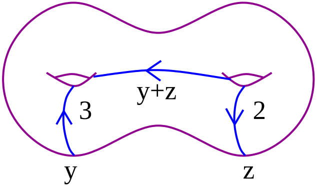

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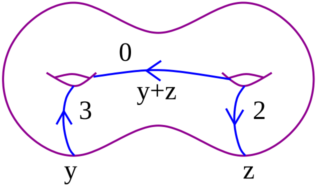

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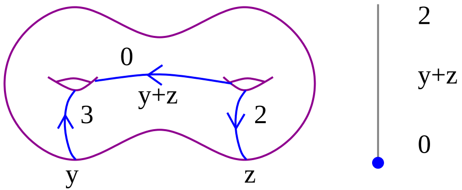

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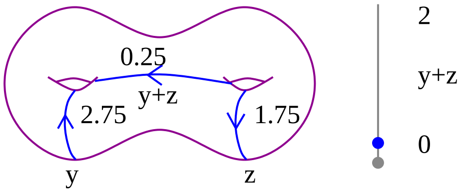

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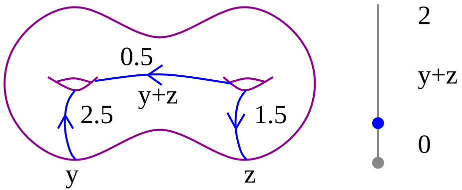

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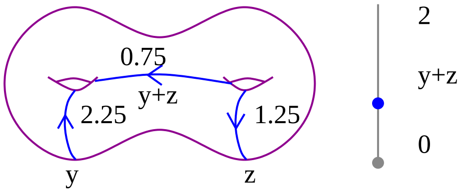

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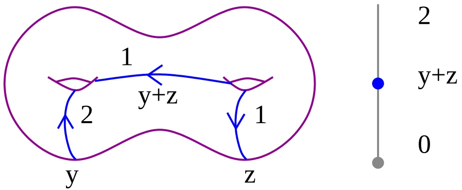

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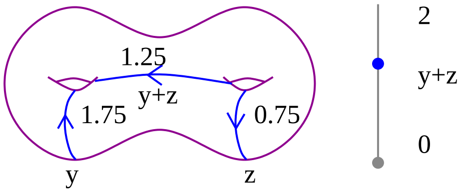

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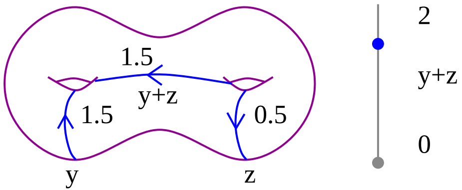

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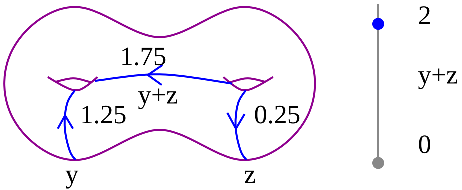

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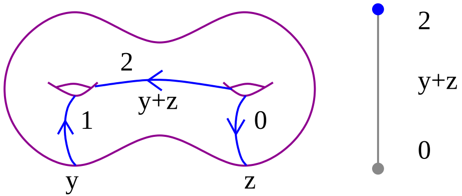

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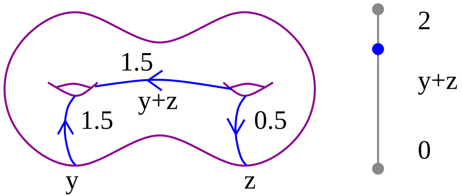

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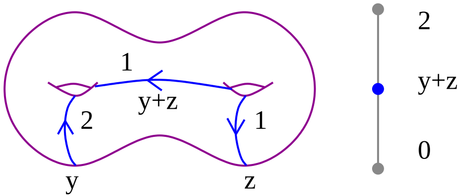

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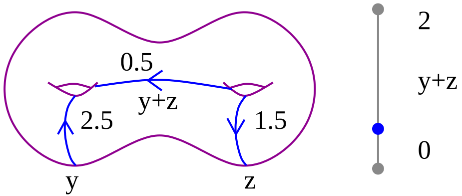

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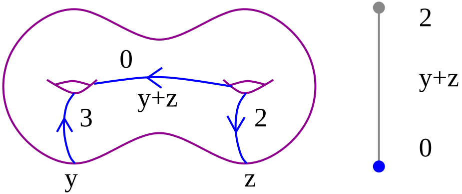

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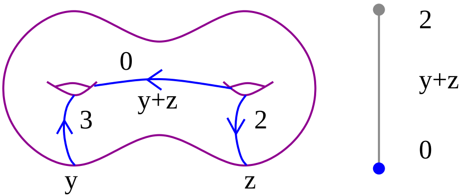
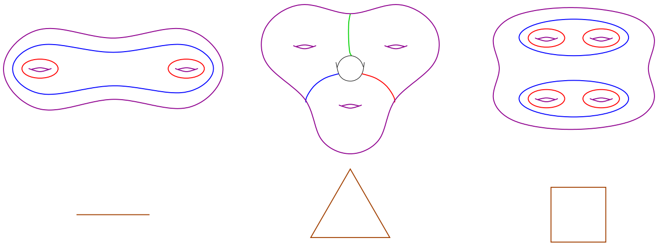


Resulting cell

Examples of cells



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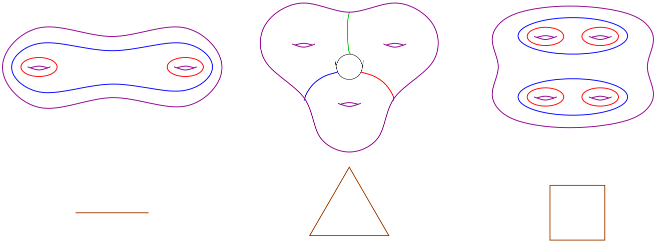

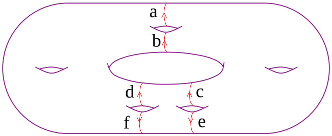

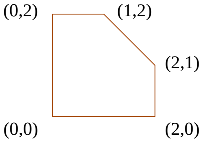

$$x = [d] + 2[e] + [f]$$

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Fact: Cell(M) is a polytope.

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Theorem (BBM)

$\mathcal{B}(S_g)$ *is contractible.*

The Complex of Cycles

Contractibility

Idea: Build analogy with Teichmüller space $\mathcal{T}(S)$.

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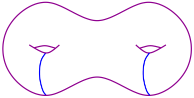

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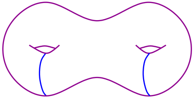
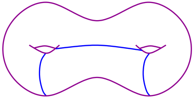


and stabilizers of edges are trivial (0-dimensional).



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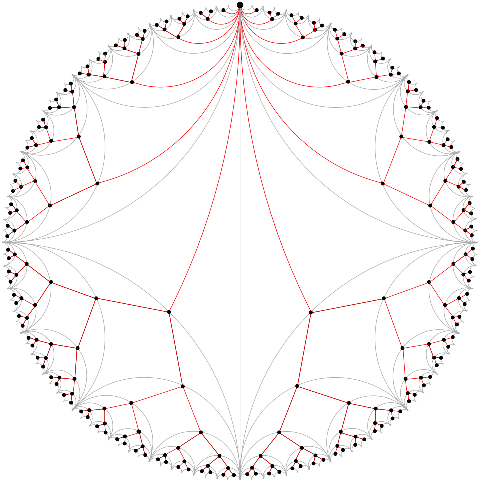

glued along their distinguished vertices.

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Therefore, to prove that (H_1 of) $\mathcal{I}(S_2)$ is infinitely generated, we just need to show that H_1 of some vertex stabilizer is infinitely generated.