# RANK OF THE FUNDAMENTAL GROUP AND HEEGAARD GENUS OF THICK NON-HAKEN 3-MANIFOLDS

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## 1. Preliminary remarks

Throughout, we let $M$ be a closed, orientable, hyperbolic 3-manifold. We define the rank of the fundamental group, $rk\,\pi_1(M)$ to be the minimal number of generators.

**Problem 1.1.** *Given $rk\,\pi_1(M)$, what can be said about $M$?*

Mostow's rigidity theorem tells us that $\pi_1(M)$ determines $M$ completely. How does this happen?

**Conjecture 1.2.** *For all $k$, there exists a $g$ such that if $rk\,\pi_1(M) = k$, then $M$ has a genus $g$ Heegaard splitting.*

This is known as the Waldhausen conjecture if $g$ is taken to be $k$, and it is known to be false in general. Zieschang has produced Seifert fiber spaces that witness the failure of the conjecture in this case.

We recall that $M = H_1 \cup_f H_2$ is a Heegaard splitting for $M$ if $H_1$ and $H_2$ are handlebodies and $f$ is a homeomorphism $\partial H_1 \to \partial H_2$.

One consequence of the conjecture is that for all $k$ there is an $A$ such that if $M$ satisfies $rk\,\pi_1(M) \leq k$ then $M$ contains an embedded surface $\Sigma$ of area less than $A$.

## 2. Results

Unless otherwise noted, we assume all manifolds $M$ satisfy $rk\,\pi_1(M) = k$. By $inj\,M$ we mean the injectivity radius of $M$. We use the term $\epsilon$-thick to mean $inj\,M \geq \epsilon$.

**Theorem 2.1** (Agol)**.** *For all $\epsilon > 0$ there exists a $V$ such that if $rk\,\pi_1(M) = 2$, $inj\,M \geq eps$, and $vol(M) \geq V$, then $M$ has a genus 2 Heegaard splitting.*

The assumptions on the injectivity radius are necessary because the proofs use limit machinery that fails if the injectivity radius is zero. The assumptions on the volume then combine with the injectivity radius to imply that there are only finitely many exceptions to the theorem for any fixed $\epsilon$.

Recall that a 3-manifold is called Haken if there exists an embedded surface $\Sigma^2$ not homeomorphic to the sphere with $\pi_1(\Sigma) \hookrightarrow \pi_1(M)$.

**Theorem 2.2** (Biringer-Souto)**.** *For all $\epsilon, k$ there exists a $g_{\epsilon,k}$ such that if $M$ is non-Haken, $inj(M) \geq \epsilon$, then $M$ has a genus $g_{k,\epsilon}$ Heegaard splitting.*

The goal of the lecture is to explain the key ideas in the proof of Agol's theorem and the differences between them and those in the proof of the Biringer-Souto theorem.

**Theorem 2.3.** *If $M_i$ is a sequence of $\epsilon$-thick hyperbolic non-Haken manifolds, then there are basepoints $p_i \in M_i$ such that up to passing to a subsequence, the manifolds $(M_i, p_i)$ converge geometrically to a tame manifold $M_G$ in such a way that $M_i$ is obtained from $M_G$ by attaching handlebodies.*

Take a sequence of such manifolds. Convergence says that compact pieces of $M_G$ can be embedded in $M_i$ for sufficiently large $i$ without distorting the metric too much. Recall that a manifold is called tame if there exists a compact $\overline{M_G}$ such that $M_G \cong int\,\overline{M_G}$. According to the work of Agol, Calegari and Gabai, this will be true if and only if $\pi_1(M)$ is finitely generated. For the second statement in the theorem, if $K_i$ is a compact exhaustion of $M_G$, the complement of the embedding of the compact pieces in the $M_i$ are handlebodies.

By looking at Heegaard splittings of $M_G$, it is possible to get a proof of Biringer-Souto.

*Indication of the proof of theorem 2.3.* If $rk\,\pi_1(M) = k$ and $F_k$ is the free group on $k$ generators, then $F_k \to \pi_1(M)$ surjectively. If $X$ is a graph, a map $X \to M$ is a carrier graph if it induces a surjection on $\pi_1$. There exists a carrier graph $X \to M$ of minimal length with $rk\,\pi_1(X) = k$. Several things can be said about $X$:

$X$ is geodesic.

$X$ is trivalent. This follows from the observation that a multivalent ($\geq 4$) graph has longer length than a trivalent one.

The angles at all the vertices are $2\pi/3$. This follows by the same reasoning as the previous point.

**Lemma 2.4.** *If $X \to M$ is a minimal length carrier graph, then one edge of $X$ is short.*

*Proof.* Assume not. Then, we can lift the map $X \to M$ to a map of universal covers $\tilde{X} \to \mathbb{H}^3$, where $\tilde{X}$ is a tree. Then, $\tilde{X} \to \mathbb{H}^3$ is a quasi-isometric embedding, which implies that $\pi_1(X) \hookrightarrow \pi_1(M)$, so that in particular $\pi_1(M)$ is free, a contradiction. □

**Lemma 2.5.** *$X$ has a short circuit.*

**Theorem 2.6** (White)**.** *For all $k$, there exists an $\epsilon_k$ such that if $rk\,\pi_1(M) = k$ then $inj\,M < \epsilon_k$.*

**Lemma 2.7** (Delzant)**.** *$X$ contains a short graph $Y \subset X$ with $\pi_1(Y)$ not abelian.*

**Theorem 2.8** (Delzant)**.** *If $G$ is Gromov hyperbolic, then there are only finitely many classes of subgroups $H < G$ with $rk\,H = 2$, $H \neq F_2$ and $H \neq \mathbb{Z}$.*

$X \to M$ a minimal length carrier graph, if $Y \subset X$ is a connected subgraph, $l_Y(X)$ is defined to be the length of $X$ outside of the (thick) convex hull of $Y$. We have that $Y \subset X \to M$ lifts to $\tilde{Y} \subset \tilde{X} \to M$

**Lemma 2.9.** *There exists a sequence* $L_k$ *such that if* $X \to M$ *is a minimal length carrier graph then there exists a sequence* $Y_0 \subset Y_1 \subset \cdots \subset Y_r = X$ *such that* $l_{Y_i}(Y_{i+1}) \leq L_k$ *for all* $i$*.*

Agol's proof uses Delzant's lemma. Assume that the $M_i$ all have $inj\ M_i \geq \epsilon$ and are non-Haken. Take $X_i \subset M_i$ a minimal length carrier graph, and for each $i$ let $Y_i^0 \subset Y_i^1 \subset \cdots \subset X_i$ be given by lemma 2.9. Up to passing to a subsequence, we may assume that while $l_{Y_i^0}(Y_i^1)$ is bounded, $l(Y_i^1) \to \infty$, and that all these graphs are homeomorphic.

Take a basepoint $p_i^0 \in Y_i^0$, and we have a limit $(M_i^0, p_i^0) \to (M_G^0, p_G^0)$

**Lemma 2.10.** $M_G^0$ *has finitely generated* $\pi_1$ *and every end of* $M_G^0$ *is degenerate.*

In particular, $M_i$ contains, for large $i$, a very large product region.

**Theorem 2.11.** *For all* $\epsilon, g$ *there exists an* $L$ *such that if* $inj\, M \geq \epsilon$*,* $M$ *does not contain incompressible surfaces of genus* $\leq g$ *and* $S \subset M$ *is a genus* $G$ *surface in a product region of width* $2$*, then* $S$ *bounds a handlebody* $H$*. Either* $\pi_1(H) \hookrightarrow \pi_1(M)$*, or* $M \setminus H$ *is a handlebody or a twisted interval bundle.*

□