Cannon-Thurston Maps and Local Connectivity of Limit Sets

Statement Of Results:

1) Connected limit sets of f.g. (3d) Kleinian groups are locally connected

2) There exist Cannon-Thurston maps for f.g. (3d) Kleinian groups. i.e. if Γ is the Cayley graph of a f.g. Kleinian group G, then (fixing a base point $0 \in \mathbb{H}^3$) the natural map $i : \Gamma \rightarrow \mathbb{H}^3$ extends continuously to a map $\hat{i} : \widehat{\Gamma} \rightarrow \widehat{\mathbb{H}^3}$ between the compactifications (interpreted appropriately for the case with parabolics)

Reduction to surface groups:

1) Anderson-Maskit (On the local connectivity of limit sets of Kleinian groups) for problem (1)

2) (M–) (Cannon-Thurston maps for Pared Manifolds of Bounded Geometry) for problem (2) (also Klarreich for case w.o. parabolics)

Motivation and History

Local Connectivity:

1) From Classical Topology: Parametrization by $[0,1]$

2) From Complex Analysis: Caratheodory extension

3) Local connectivity of Julia sets and Sullivan-McMullen dictionary

Cannon-Thurston Maps:

Theorem ELC (Minsky, Masur-Minsky, Brock-Canary-Minsky) Simply degenerate hyperbolic manifolds homeomorphic to $S \times (-\infty, \infty)$ are determined by two pieces of data:

1) The conformal structure on the boundary S

2) the ending lamination for the simply degenerate end.

A putative characterization in terms of action of $\pi_1(S)$ on the sphere at infinity:

1) The conformal structure on the boundary S

2) the action of the surface group on its limit set.

Connection between these two is forged by the following.

Question (Cannon-Thurston): Suppose a closed surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries. Does the inclusion $\tilde{i} : \widetilde{S} \to \mathbb{H}^3$ extend continuously to the boundary?

For a simply degenerate group, this is equivalent to asking if the limit set is locally connected.

In fact, we show that:

Theorem: Limit set is the compactified R-tree dual to the ending lamination.

STRATEGY

I) Technique for reduction to surface groups and handling parabolics (CT for pared manifolds of bdd geo)

II: Model Geometries

(IIa)

1) Bounded Geometry

2) i-bounded Geometry

(IIb) 3) Amalgamation Geometry

4) Weak Amalgamation Geometry

5) Split Geometry

III

6) All 3-manifolds have split geometry (uses Minsky's model for ELC)

7) Epilogue: Parallels with Complex Dynamics

Part I: Model Geometries:

Bounded Geometry: History, several cases

Theorem (Cannon-Thurston, 1989): Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. The inclusion $\tilde{i} : \widetilde{F} \to \widetilde{M}$ extends to a continuous map $\hat{i} : \mathbb{D}^2 \to \mathbb{D}^3$.

Cannon-Thurston's original proof involves construction of singular metric from foliations (cf Ian Agol's course at UCB).

1) Minsky ('94): bounded geometry closed surface groups

2) M–('98), Klarreich('99): Bounded geometry 3 manifolds with incompressible core and *no parabolics*

3) Miyachi ('02): Bounded geometry 3 manifolds with (possibly) compressible core and *no parabolics* (see also Souto ('06))

4) Alperin-Dicks-Porti ('99): Fig8 Knot complement;

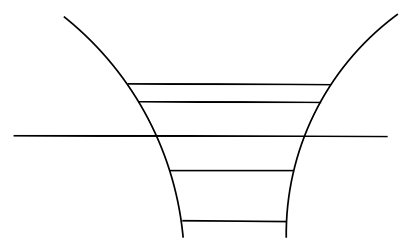
Bowditch ('02): bounded geometry punctured surface groups

5) M– ('05): Cannon-Thurston Maps for Pared Manifolds of Bounded geometry: Technique to reduce to surface groups, and handle parabolics

(1.2) Alternate Proof - Bounded Geometry (M–)

Equispaced Stack of hyperbolic planes (sheets)

Uniform QI from one sheet to the next

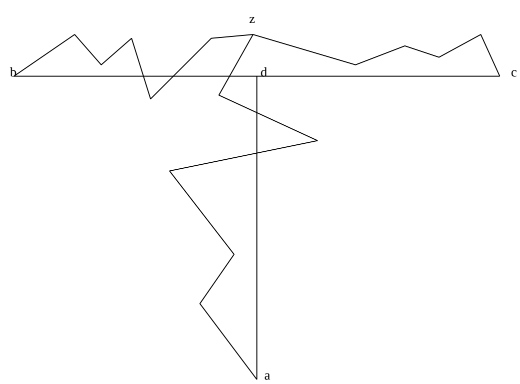


Crucial Steps:

1) Construct *Hyperbolic Ladder* $B_\lambda = \bigcup_i \lambda_i$

2) Construct Sheetwise projection $\Pi_\lambda = \bigcup_i \pi_i$

3) Quasi-isometries and Nearest Point Projections almost commute



4) Hence B_λ uniformly quasiconvex

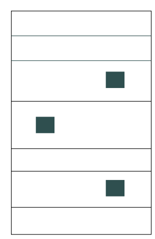
5) Information on Hyperbolic geodesic from quasiconvexity of B_λ

(2) i-bounded Geometry

Special Case: Punctured Torus (McMullen, 2001) using Minsky Model.

Still have a stack of hyperbolic surfaces bounding regions that contain Margulis tubes, whose boundary has bounded diameter

Bounded geometry away from Margulis tubes



Problems

1) Quasi-isometry from one sheet to the next - not uniform.

2) Successive sheets not equispaced

Proposed Solution

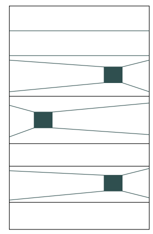
1) Electrocute Margulis tubes.

2) Dehn twists are electric isometries Hence, QI in the electrocuted metric is now uniform

3) Stack of hyperbolic planes (sheets) with electrocuted torii - equispaced away from Margulis tubes. Hence equispaced after electrocution.

4) Regain information about hyperbolic metric from electrocuted metric

(3) Amalgamation Geometry "Dual to" i-bounded Geometry



Thick blocks as before

Thin blocks replaced by *Amalgamation blocks*

Each amalgamation block has one or more Margulis tubes

(Lifts of) Complementary components plus bounding Margulis tubes uniformly quasiconvex in $\widetilde{M}$

These components will be called *amalgamation components in* $\widetilde{M}$

The intersection of these with horizontal sheets $\widetilde{S}$ will be called *amalgamation components in* $\widetilde{S}$

(3a) Weak Amalgamation Geometry

This is a slight weakening of *amalgamation geometry*.

We require that the convex hulls $CH(K)$ of amalgamation components K lie in a uniformly bounded neighborhood of K in $(\widetilde{M}, d_G)$ rather than $(\widetilde{M}, d_{hyp})$.

So the K's are uniformly *graph-quasiconvex* rather than *hyperbolic* quasiconvex.

5-holed sphere groups have weak amalgamation geometry.

(4) Split Geometry

(4.1) Split Surfaces and Split Blocks

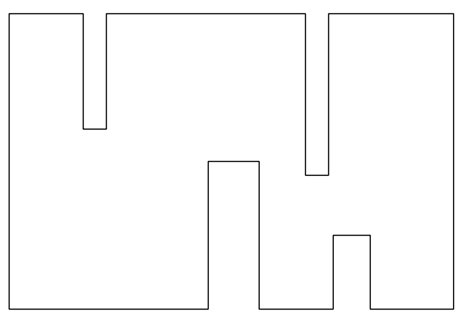
Toplogically, a **split subsurface** S^s of a surface S is a (possibly disconnected, proper) subsurface with boundary such that $S - S^s$ consists of a non-empty family of non-homotopic annulii, which in turn are not homotopic into the boundary of S^s.

Geometrically, we assume that S is given some finite volume hyperbolic structure. A split subsurface S^s of S has bounded geometry, i.e.

a) each boundary component of S^s is of length ϵ_0, and is in fact a component of the boundary of $N_k(\gamma)$.

b) For any closed geodesic β on S, either $\beta \subset S - S^s$, or, the length of any component of $\beta \cap (S^s)$ is greater than ϵ_0.

Split surfaces bound *split blocks*
Upper and lower boundaries of split blocks are split subsurfaces of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**.



(4.2) Tube Electrocution for Split Blocks

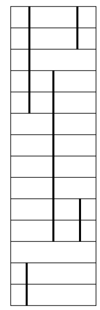
Welded split block = split block with identifications as follows: Components of $\partial S^s \times 0$ are glued together if and only if they correspond to the same geodesic in $S - S^s$. The same is done for components of $\partial S^s \times 1$. A simple closed curve that results from such an identification shall be called a **weld curve**. For hanging tubes, we also weld the boundary circles of their *lower or upper boundaries* by simply collapsing $S^1 \times [-\eta, \eta]$ to $S^1 \times \{0\}$. Also record *twist information* about *weld curves*. Let the metric product $S^1 \times [0,1]$ be called the **standard annulus** if each horizontal S^1 has length ϵ_0. For hanging tubes the standard annulus will be taken to be $S^1 \times [0, 1/2]$.

Another pseudometric on B – **tube-electrocuted metric**.

Map from each boundary annulus $S^1 \times I$ (or $S^1 \times [0, 1/2]$ for hanging annulii) to the corresponding standard annulus that is affine on

the second factor and an isometry on the first. Glue the mapping cylinder of this map to the boundary component. The resulting 'split block' has a number of standard annulii as its boundary components. Call the split block B^s with the above mapping cylinders attached, the *stabilized split block* B^{st}.
Glue boundary components of B^{st} corresponding to the same geodesic together via 'identity map' to get the **tube electrocuted metric** on B by putting the zero metric on each resulting $S^1 \times \{x\}$.
Annulus $S^1 \times J$ obtained via this identification has the zero metric in the *horizontal direction* $S^1 \times \{x\}$ and the Euclidean metric in the *vertical direction* J.
The resulting block will be called the **tube-electrocuted block** B_{tel} and the pseudometric on it will be denoted as d_{tel}.

(4.3) Weak Split Geometry



A manifold $S \times \mathbb{R}$ equipped with a vertical Margulis tube structure $\mathcal{T}$ (adapted from vertical annulus structure, cf Bowditch) is said to be a model of **weak split geometry**, if it is equipped with a new metric satisfying the following conditions:
a) $S \times [m, m+1] \cap Int\mathcal{T} = \emptyset$ (for $m \in \mathbb{Z} \subset \mathbb{R}$) implies that $S \times [m, m+1]$ is a thick block.
b) $S \times [m, m+1] \cap Int\mathcal{T} \neq \emptyset$ (for $m \in \mathbb{Z} \subset \mathbb{R}$) implies that $S \times [m, m+1] - Int^{+}\mathcal{T}$ is (geometrically) a split block.
c) There exists a uniform upper bound on the lengths of vertical intervals for vertical Margulis tubes
d) The metric on each component Margulis tube T of $\mathcal{T}$ is hyperbolic

(4.4) Definition of Split Geometry
For each split component $\tilde{K}$ construct $\tilde{K} \times [0, \frac{1}{2}]$ and put zero metric on $\tilde{K} \times \{\frac{1}{2}\}$. The metric on the resulting space shall be called the **graph metric** d_G on $\tilde{M}$.

Definition: A split amalgamation component K is said to be **(m. κ) - graph quasiconvex** if there exists a κ-quasiconvex (in the hyperbolic metric) subset $CH(K)$ containing K such that
1) $CH(K) \subset N_m^G(K)$ where $N_m^G(K)$ denotes the m neighborhood of K in the graph metric d_G on M.
2) For each K there exists C_K such that K is C_K-quasiconvex in $CH(K)$.

Definition: A model manifold M of weak split geometry is said to be a model of **split geometry** if there exist m, κ such that each split amalgamation component is (m, κ) - graph quasiconvex.

Part II: Minsky Model and Split Geometry

We construct from the Minsky model the following:

1) A sequence of split surfaces S_i^s exiting the end(s) of M. These will determine the levels for the split blocks and split geometry. There has to be a lower bound on the distance between S_i^s and S_{i+1}^s

2) A collection of Margulis tubes $\mathcal{T}$.

3) For each complementary annulus of S_i^s with core σ, there is a Margulis tube T whose core is freely homotopic to σ and such that T intersects the level i. (What this roughly means is that there is a T that contains the complementary annulus.)

4) For all i, either there exists a Margulis tube splitting both S_i^s and S_{i+1}^s and hence B_i^s, or else B_i is a thick block.

5) $T \cap S_i^s$ is either empty or consists of a pair of boundary components of S_i^s that are parallel in S_i.

6)There is a uniform upper bound n on the number of surfaces that T splits.

In addition to prove that the Minsky model gives rise to a model of *split geometry*, we show that each split component is

7) (not necessarily uniformly) quasiconvex in the hyperbolic metric
8) uniformly graph-quasiconvex

Theorem: The Minsky model corresponding to a simply or doubly degenerate surface group has split geometry.

(7) Extending the Sullivan-McMullen Dictionary

1) J_f - Julia set $\leftrightarrow$ Λ_Γ - Limit Set

2) Equipotential curve - γ_0 $\leftrightarrow$ Base split surface $S_0, \widetilde{S_0}$

3) Equipotential curve - γ_i $\leftrightarrow$ Split surface $S_i, \widetilde{S_i}$

4) Landing rays r_{ij} cutting up the region between γ_i and J_f $\leftrightarrow$

in the split geometry case, thin Margulis tubes splitting the split block B_i^s

5) Puzzle pieces U_{ij} $\leftrightarrow$ Split components $\widetilde{K}$ in the universal cover $\widetilde{M}$

6) Vanishing of nested puzzle pieces U_{ij} as $i \to \infty$ $\leftrightarrow$ Uniform graph quasiconvexity of split components.