Positivity of the universal pairing in 3 dimensions, and the topological Cauchy-Schwarz inequality

Danny Calegari (joint w/ Mike Freedman and Kevin Walker)

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Atiyah: an $n+1$ dimensional TQFT is a functor

$$Z : (n\text{-manifolds}, \text{cobordisms}) \to (\mathbb{C}\text{-vector spaces}, \text{linear maps})$$

satisfying

$$Z(S_1 \coprod S_2) = Z(S_1) \otimes Z(S_2)$$

It follows that $Z(\emptyset) = \mathbb{C}$. If $\emptyset \xrightarrow{A} S$ is a cobordism,

$$Z(A) = Z(A)(1) \in Z(S)$$

by abuse of notation. Assume images $Z(A)$ span $Z(S)$. There is a pairing

$$Z(S) \times Z(\overline{S}) \to Z(\emptyset) = \mathbb{C}$$

defined on generators by

$$\emptyset \xrightarrow{A} S \circ S \xrightarrow{B} \emptyset = \emptyset \xrightarrow{AB} \emptyset$$

Physical axiom: pairing is *unitary* ($Z(A\overline{A}) = \|A\|^2 > 0$)

There are many interesting unitary TQFT's e.g. $SU(2)$ Chern-Simons, Jones polynomial, etc.

Fundamental question: what kind of information (in a given dimension) can be extracted (in principle) from unitary TQFT's?

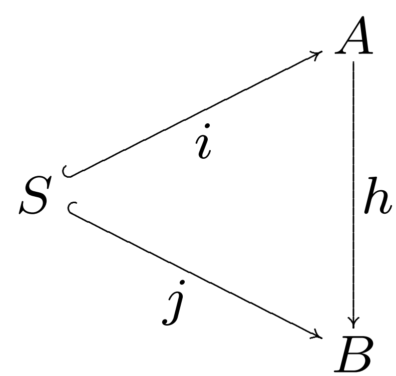
We can address this issue by studying a *universal* object, through which all unitary TQFT's (in a given dimension) must factor. This universal object makes sense in any dimension; we will focus on the case of $2+1$ dimensions.

S is a closed, oriented surface.

$\dot{\mathcal{M}}(S) :=$ the set of equivalence classes of pairs $\{(A, S)\}/\sim$

where A is a compact, oriented 3-manifold with $\partial A = S$, and

$(A, S) \sim (B, S)$ if there is a commutative diagram



Special case: $\dot{\mathcal{M}} := \dot{\mathcal{M}}(\emptyset)$

$\mathcal{M}(S) :=$ $\mathbb{C}$-vector space spanned by the set $\dot{\mathcal{M}}(S)$

$\mathcal{M} := \mathcal{M}(\emptyset)$

If $A, B \in \dot{\mathcal{M}}(S)$, AB is the result of gluing A to $\overline{B}$ along S.

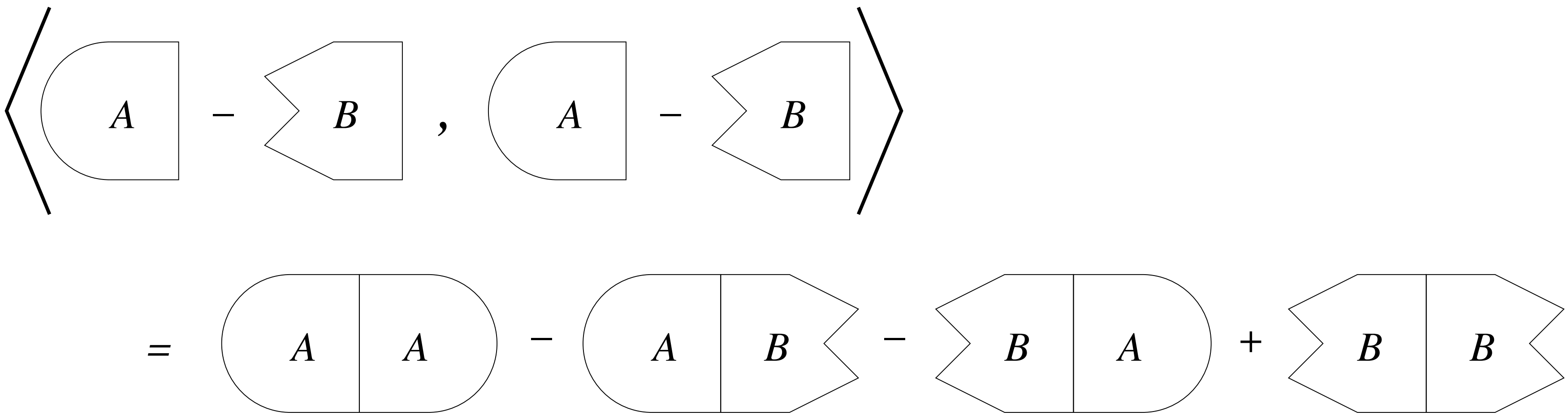
The *universal pairing* is the map

$$\langle \cdot, \cdot \rangle_S : \mathcal{M}(S) \times \mathcal{M}(S) \to \mathcal{M}$$

defined by the formula

$$\left\langle \sum_i a_i A_i, \sum_j b_j B_j \right\rangle = \sum_{i,j} a_i \bar{b}_j A_i B_j$$

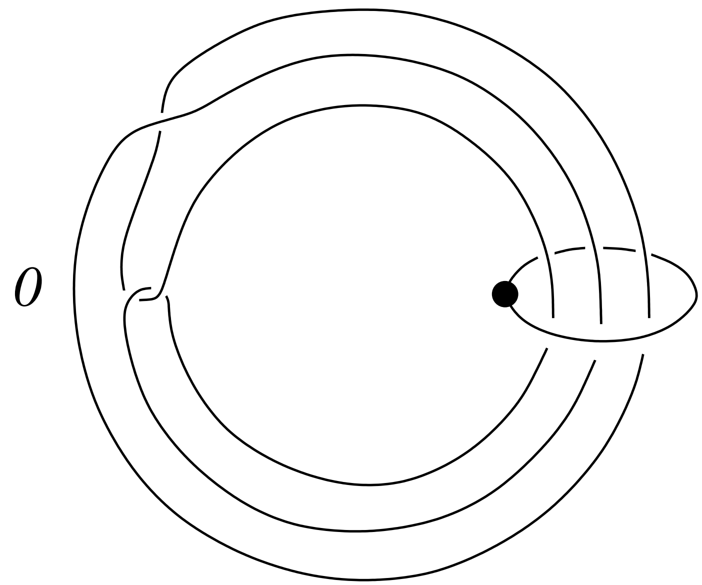
Example: $\langle A - B, A - B \rangle = AA - AB - BA + BB$



$-1+1$, $0+1$, $1+1$ dimensions: positivity holds. (easy exercise)

$\geq 3+1$ dimensions: positivity *fails*! (in smooth category) In fact, all interesting smooth 4-manifold topology is in the kernel! (Freedman-Kitaev-Nayak-Slingerland-Walker-Wang in dimension 4, Kreck-Teichner in dimension ≥ 5)

Example. M Mazur manifold.



There is an involution $\theta : S \to S$ which does not extend *smoothly* over M, so $M, \theta(M)$ are distinct elements of $\dot{\mathcal{M}}(S)$. But

$$M\overline{M} = M\overline{\theta(M)} = \theta(M)\overline{M} = \theta(M)\overline{\theta(M)} = S^4$$

so $x = M - \theta(M) \in \mathcal{M}(S)$ is nonzero, with $\langle x, x \rangle = 0$.

Main result: positivity holds in $2+1$ dimensions!

Theorem (Positivity). *For any S and any $v \in \mathcal{M}(S)$,*

$$\langle v, v\rangle_S = 0 \text{ if and only if } v = 0$$

Thus, in principle, unitary $2+1$ dimensional TQFT's should be sensitive to a great deal of 3-manifold topology.

How to prove positivity in 2+1 dimensions? Use Cauchy-Schwarz inequality! (topologist style)

Theorem. *There is a complexity function* $c : \dot{\mathcal{M}} \to \mathcal{O}$ *for some ordered set* $\mathcal{O}$ *so that for all* $A, B \in \dot{\mathcal{M}}(S)$*,*

$$c(AB) \leq \max(c(AA), c(BB))$$

with equality if and only if $A = B$.

How does this prove theorem? Write $v = \sum_i a_i A_i$ where $a_i \in \mathbb{C}^*$ and $A_i \neq A_j$ when $i \neq j$. Expand:

$$\langle v, v \rangle = \left\langle \sum_i a_i A_i, \sum_i a_i A_i \right\rangle = \sum a_i \bar{a}_j A_i A_j$$

Let i, j be such that $N := A_i A_j$ maximizes the complexity (in these factors). Then by topological CS-inequality, we must have $i = j$. So the coefficient of N is a sum of terms of the form $a_i \bar{a}_i = \|a_i\|^2$ which is positive. In other words, the coefficient of N is nonzero, so $\langle v, v \rangle \neq 0$ in $\mathcal{M}$.

It remains to define c.

The complexity function c is itself quite complicated, because the (geometric) classification theorem for 3-manifolds has a *hierarchical structure*:

A closed 3-manifold is decomposed into its connected components.

A closed, connected 3-manifold is decomposed into its prime factors.

A closed, connected, prime 3-manifold has a JSJ decomposition into (possibly noncompact) Seifert fibered and hyperbolic pieces or spherical space forms.

The complexity function c has to have terms which treat each of these factors!

$$c = (c_0, c_1, c_2, c_3)$$

where c_0 treats connectivity, c_1 treats the kernel of inclusion $\pi_1(S) \to \pi_1(A)$, c_2 treats essential 2-spheres, c_3 treats prime factors.

c_3 complicated function of factors, using a further term c_p.

c_p complexity function on *prime* closed 3-manifolds so that if A, B are connected, orientable, irreducible, boundary irreducible, with incompressible boundary S, then $c_p(AB) \leq \max(c_p(AA), c_p(BB))$ with equality if and only if $A = B$.

$$c_p = (c_{SF}, c_h, c_a)$$

where c_{SF} treats Seifert fibered pieces, c_h treats hyperbolic pieces, c_a treats way in which these pieces are assembled together (has further subterms).

c_{SF}, c_a complicated, but in some ways similar to lower dimensional case.

$$c_h = (-\text{vol}(M), \sigma(M))$$

where vol is hyperbolic volume, σ is geodesic length spectrum (with multiplicity) of geodesics whose complex length is real, ordered alphabetically (more short geodesics is better).

Heart of hyperbolic case is:

Theorem. *S orientable surface of finite type, negative Euler characteristic, A, B irreducible, atoroidal and acylindrical with boundary S. Then AA, AB, BB admit unique complete hyperbolic structures, and*

$$2\mathit{vol}(AB) \geq \mathit{vol}(AA) + \mathit{vol}(BB)$$

with equality if and only if S is totally geodesic in AB.

(generalizes closed case due to Agol-Storm-Thurston).

Find least area surface $S \subset AB$.

Cut open to get A, B, and double metrically to get AA, BB with C^0 Riemannian metrics.

Minimal surface = vanishing mean curvature
= flat on average, to second order.

After doubling, volume growth rate = rate in hyperbolic space, to second order. I.e. (distributional) scalar curvature on AA, BB is identically -6.

Hamilton's equation for scalar curvature for normalized flow (i.e. Ricci flow rescaled to preserve volume):

$$\frac{dR}{dt} = \Delta R + 2|\mathrm{Ric}_0|^2 + \frac{2}{3}R(R - r)$$

where r is spatial average of R.

Under normalized flow, spatial infimum $\check{R}$ of R (if negative) is nondecreasing, and *strictly increasing* unless metric is hyperbolic (slightly more complicated in noncompact case).

For fixed volume, infimum of scalar curvature increases. Rescale so that infimum of scalar curvature is $\equiv -6$, volume *decreases*.

Perelman: at singularities of the flow, $R \to +\infty$, so can do surgery without changing $\check{R}$; moreover, long time flow converges to hyperbolic metric. Hence volume of hyperbolic metric is *less* than volume of "doubled" metric.

Technical issue: Hamilton's equation is for *smooth* metrics; but doubled metrics are just C^0.

Miao, Bray: metrics can be approximated by C^∞ metrics with $R \geq -6$ pointwise (by mollifying, or inserting plugs).

Miles Simon: certain C^0 metrics can be Ricci flowed (short time existence), and distributional $R \geq -6$ is preserved infinitesimally

Technical issue: Simon requires a background metric with uniform curvature bounds, which is $1+\epsilon$ bilipschitz to the C^0 metric for some fixed (but possibly very small) ϵ.

In the closed case (AST) uniform curvature bounds are free by compactness.

We need to find a uniform bilipschitz model for the doubled metric in the cusps.

Lemma. *Least area surfaces in cusps of hyperbolic* 3*-manifolds become asymptotically flat* faster *than the thickness of the cusp goes to* 0*.*

Hence the ends of AA, BB with the doubled metric are asymptotically $1 + \epsilon$ bilipschitz to a *hyperbolic* metric, for any $\epsilon > 0$.

Idea: a least area surface in a 3-manifold with enough symmetry is totally geodesic. The parabolic stabilizer of the cusp of the surface looks more and more like a continuous family of isometries deeper and deeper in the cusp.

If S is totally geodesic, hyperbolic case is completed by

Theorem. *Suppose A, B are hyperbolic with totally geodesic boundary S so that the induced metrics on S are isometric. Then*

$$\sigma(AB) \leq \max(\sigma(AA), \sigma(BB))$$

with equality if and only if $A = B$.

Idea of proof: look at set of geodesic arcs in A and B perpendicular to S, ordered by length, and match them one by one inductively.