COUNTING PERIODIC TECHMÜLLER GEODESICS IN ODD STRATA

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1. Advertisement

There is a new journal coming soon: *Analysis and PDE*. Information can be found at http://pjm.math.berkeley.edu/apde/about/cover/cover.html.

2. Setup

Let S be a closed surface, $g \geq 2$. Let $T(S)$ be Teichmüller space determined by S and $Q^1(S)$ be the holomorphic quadratic differentials of area 1. The strata are defined by considering those differentials vanishing to order $\overline{k}$, where $\overline{k} = (k_1, \ldots, k_l)$, and $\sum k_i = 4g-4$. This is a complex submanifold of dimension $h = 2g+l-2$ and is invariant under the action of the mapping class group, hence projects to a stratum in moduli space.

Many things are known about strata. Masur-Smilie show that the strata are nonempty. In $'03$, Kontsevich-Zorich, and later Lanneau in $'06$ showed that the strata are generally not connected, and the components can be described combinatorially. Teichmüller flow acts on components of strata. Masur-Veech showed that there exist invariant probability measures on $Q(component)$ of Lebesgue measure class.

Let $\gamma \subset Q$ be a periodic orbit of the Teichmüller flow of prime period $l(\gamma)$. This defines a flow-invaritant measure on Q of total mass $l(\gamma)$. If Q is a stratum, and if $\Gamma(R)$ is the set of periodic orbits of length $\leq R$ in Q, then the measure is given by

$$\mu_R = he^{-hR} \sum_{\gamma \in \Gamma(R)} \delta(\gamma),$$

where the h above is the same as before.

Theorem 2.1 (Labor)**.** *μ_R converges weakly to Lebesgue measure as $R \to \infty$. We have no control on the asymptotic growth of periodic orbits.*

Corollary 2.2. *As $R \to \infty$, $\liminf(\#\{\gamma \in \Gamma(R) | l(\gamma) \leq R\} := n_Q(R)) \geq e^{hR}/hR$.*

Theorem 2.3 (Art, Eskin-Mirzakhani)**.** *Let $Q = Q(1, \ldots, 1)$, $n_Q(R)hRe^{-hR} \to 1$ as $R \to \infty$, and for all $\epsilon > 0$, there exists a compact $K \subset Mod(S)$ such that*

$$\limsup_{R\to\infty} \frac{1}{R} \log \#\{\gamma | \gamma \cap K = \emptyset\} \leq h - 1 + \epsilon.$$

3. Main results

We say $Q(\overline{k})$ is odd if each k_i is odd.

Theorem 3.1. *If $Q \subset Q(k_1, \ldots, k_l)$ is odd, then $n_Q(R)hRe^{-hR} \to 1$.*

Sketch of proof. First, we say that $q \in Q$ is non-diverging if there exists a compact $K \subset Q(S)$ such that $\phi^t(q)$ returns to K for arbitrarily large or small t. According to Masur, these are uniquely ergodic, i.e. the horizontal and vertical folitations are uniquely ergodic.

The curve graph is the 1-skeleton of the curve complex. According to Masur-Minsky, this is hyperbolic with the graph metric. We define a map $Y : T(S) \to \mathcal{C}(S)$ that takes a point x to a simple closed geodesic of length $\mathcal{H}$, the Bers constant. Masur-Minsky says that if γ is a Teichmüller geodesic, then its image under Y is an unparametrized p-quasi-geodesic.

Define $\tilde{A}$ to be $q \in Q^1(S)$ such that $t \to Y(p\phi^t q)$ has infinite diameter. For $q \in \tilde{A}$, let $F(q)$ be the endpoint of $Y(p\phi^t q)$, which lies in the Gromov boundary of the curve complex.

Lemma 3.2. *If $\tilde{A}$ is the set of q the fill up S, then $\tilde{A}$ is a G_δ subset of $Q^1(S)$ that is MCG-invariant. The map F defined above is continuous, closed and MCG-equivariant. Furthermore, if $q \in \tilde{A}$ is uniquely ergodic and if $\{U_i\}$ is a neighborhood basis of q in $Q^1(S)$, then $\{F(U_i \cap \tilde{A})\}$ is a neighborhood basis of q in the Gromov boundary.*

Lemma 3.3. *If q is non-diverging, then for all ϵ and all open neighborhoods V of q, there exists a neighborhood $U \subset V$ of q such that $\mu(U) \leq \lambda(U)(1+\epsilon)$, where λ denotes Lebesgue measure.*

Corollary 3.4. *If R is the set of nondiverging points, then $\mu|_R \leq \lambda|_R$.*

Lemma 3.5. *There exists an $m > 0$ such that $\mu(\{q| \, diam(q) \geq m\} - R) = 0$.*

Next we show that $\mu(A = \{q| \, diam(q) \leq m\}) = 0$. Assuming otherwise, there exists a density point q_0 for A, which implies that the orbit of q_0 "goes straight into the cusp" and can be approximated by $\sim ce^{hl}$ periodic geodesics of length less than or equal to l. We use this to derive a contradiction to the previous lemma. Technical issues are resolved by the notion of an electric train track.

An electric train track τ is a train track on S such that every component of τ is a polygon with $k_i + 2$ sides for all i, τ admits a positive transverse measure, with $\nu(\tau)$ being the space of all of these, and τ admits a positive tangential measure μ, the space of which is denoted $E(\tau)$.

Some elementary facts about electric train tracks are as follows: each tangential measure determines a connected, compact family of measure geodesic laminations on S. Every transverse measure on τ determines uniquely a measured geodesic lamination carried by τ. For each $\nu \in \nu(\tau)$ and $\mu \in E(\tau)$ positive with $\sum \nu(b)\mu(b) = 1$, there exists a quadratic differential in $\overline{Q(\overline{k})}$ with horizontal lamination ν, vertical lamination in a family defined by μ. Taking all differentials with $\nu(\tau) = 1$, $Q(\tau)$, gives a closed subset of $\overline{Q(\overline{k})}$. If $q \in Q(\overline{k})$, then $\phi^t q \in Q(\tau)$ for some t and τ. □

Theorem 3.6. *Lebesgue measure is the unique measure of maximal entropy.*