GRAFTING RAYS AND TEICHMÜLLER GEODESICS

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1. Main results

This talk is based on work joint with Choi and Dumas.

Grafting is a map from $\mathcal{ML}(S) \times T(S) \to T(S)$ given by cutting along a simple closed geodesic of weight w and gluing in a flat cylinder. The construction extends to multicurves, and by continuity, to laminations Fix $X \in T(X)$ and $\lambda \in \mathcal{ML}(S)$.

$$gr_\lambda(S) = gr(s\lambda, X) : R_+ \to T(S)$$

is called the grafting ray.

Theorem 1.1 (Hubbard-Masur)**.** *There exists a quadratic differential q in the conformal class of X with vertical foliation equal to λ.*

We write $T(t) = [q_t]$ where q_t is given by the geodesic flow applied to q. This is called the Teichmüller ray.

Theorem 1.2. *$gr(\lambda, \cdot)$ is uniformly Lipschitz. In particular, there exists an L such that for all pairs $X, Y \in T(S)$,*

$$d_T(gr(\lambda, X), gr(\lambda, Y)) \leq L d_T(X, Y).$$

Theorem 1.3. *Grafting rays fellow travel Teichmüller geodesics. That is, for all X there is a D such that for all λ,*

$$d_T(gr_\lambda(\frac{e^{2t} - 1}{|\lambda|}), T(t)) \leq D.$$

Theorem 1.4 (Diaz-Kim)**.** *For all X and all simple closed curves λ, there is a D such that $d_T(gr(\lambda, X), T) \leq D$.*

Corollary 1.5. *Grafting rays are Teichmüller quasi-geodesics.*

Corollary 1.6. *The shadow of gr_λ to $C(S)$ is an unparametrized quasi-geodesic.*

Indication of a proof of theorem 1.3. Given a complicated lamination, we wait until $s = 1$ and then collapse the hyperbolic part into the flat part to get a singular flat structure, i.e. a quadratic differential, with some care at the infinitesimal level. □

Sketch of a proof of theorem 1.2. Let QF_k be the space of k-quasi-Fuchsian groups. Extend the grafting map to $gr : \mathcal{ML}(S) \times QF_k(S) \to T(S)$. We equip QF_k with the Kobayashi metric. Pick a Teichmüller disk D passing through X. We obtain

$$d_T(X, Y) > d_k((X, X), (Y, Y)).$$

$gr\{\lambda\} \times QF_k \to T(S)$ is a holomorphic map, so that

$$d((X,X),(Y,Y)) \geq d_T(gr_\lambda(X), gr_\lambda(Y))$$

which implies that the map is Lipschitz. □

Theorem 1.7. *There exist X_n and λ_n such that $d(gr_{\lambda_n}(X_n), T_n) \to \infty$ in the Hausdorff topology, where T_n is the corresponding Teichmüller ray.*

Now, notice that

$$T(S) \sim T(S_{1,1}) \times T(S_{1,1}) \times \mathbb{H}^2,$$

where the last factor is the gluing parameter.

Completion of the proof of theorem 1.3. Prove the theorem for a fixed X and for a small neighborhood $U \subset \mathcal{PML}(S)$. Let φ is pseudo-Anosov, and $\lambda_\varphi^\pm$ are the stable and unstable foliations, $\lambda_\varphi^+ \in U$ and $\lambda_\varphi^- \in V$ for U, V small. Assume $\lambda \notin V$, Y near X, then there exists n such that $\varphi^n(\lambda) \in U$. By theorem 1.2, $gr_{\lambda,X}$ is close to $gr_{\lambda',Y'}$. In particular, $gr_{\lambda,X}$ fellow travels a Teichmüller ray, which implies that $gr_{\lambda',Y'}$ fellow travels a Teichmüller ray, which implies that $gr_{\lambda,T}$ fellow travels a Teichmüller ray. □