SOME REMARKS ON TRANSLATION SURFACES IN GENUS 3

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This talk is based on work in progress with Erwan Lanneau and Marlin Möller.

1. Translation surfaces

We have a combinatorial definition of translation surfaces as a finite number of disjoint polygons $\{P_1, \dots, P_m\}$ in the plane, and for every side $C \in P_i$ with $C' \in P_j$ parallel and of the same length, we identify $C \sim C'$.

Such a translation surfaces have a flat metric, even on the boundary. On the boundary, we have conical singularities. The angles at these vertices are multiples of 2π, and the gluings are obtained by translation, which is holomorphic. Thus, the resulting surface has a natural Riemann surface structure. We also get a holomorphic 1-form dz that extends to the singularities.

We also have an analytic definition of translation surfaces. We start out with a holomorphic 1-form $f(z)\,dz$ on a compact Riemann surface. We get a translation structure by taking $f(z_0)\,dz \to d\omega$ locally if $f(z_0) \neq 0$. On the other hand, if $f(z_0) = 0$, we locally obtain $\omega^k\,d\omega$ if z_0 is a zero of order k. We get a singular vertex at the zero, and the angle there will be $2(k+1)\pi$. Polygons also give a translation structure, and we remark that half-translation surfaces are in correspondence with quadratic differentials.

If $k_1, \dots, k_n = \overline{k}$ are positive integers, we let $H(\overline{k})$ denote the set of translation surfaces with zeroes with precisely those orders. We have the equality $2g - 2 = \sum_i k_i$. For $g = 2$, we only get $H(2)$ and $H(1,1)$.

Henceforth we work in the moduli space, not the Teichmüller space. This is reasonable from a dynamics point of view: we look at the $SL_2\mathbb{R}$ action. If (X,ω) is defined by polygons, i.e. $(X,\omega) \leftrightarrow (P_1, \dots, P_n)/\sim$, and $A \in GL_2\mathbb{R}$, we write $A(X,\omega) = (AP_1 \dots, AP_n)/\sim$.

2. Topology and measure on the strata

If $\{A_i\}, \{B_i\}$ are a basis for the first homology of X, and $\{C_i\}$ are intervals between zeroes of the form ω, we get period coordinates that endow $H(\overline{k})$ with a complex structure:

$$\{\int_{A_i} \omega\}, \{\int_{B_i} \omega\}, \{\int_{C_i} \omega\}.$$

$dim_{\mathbb{C}} H(\overline{k}) = 2g + n - 1$. The measure on the stratum can then be given by Lebesgue measure. Let $H^1(\overline{k})$ denote the area 1 translation surfaces.

Problem 2.1. *What is the closure of the $SL_2\mathbb{R}$ orbit of (X,ω)?*

Theorem 2.2 (Masur-Veech '82)**.** *For a.e.* $(X,\omega) \in H^1(\overline{k})$*, the* $SL_2\mathbb{R}$ *orbit of* (X,ω) *is dense in the connected component of* $H^1(\overline{k})$ *containing* (X,ω)*.*

To prove this, one shows that the geodesic flow g_t is ergodic.

The Veech group of (X,ω) is the stabilizer of the $SL_2\mathbb{R}$ orbit of (X,ω). It is denoted $SL(X,\omega)$. If $(X,\omega) = (\mathbb{T}^2, dz)$, then $SL(X,\omega) = SL_2\mathbb{Z}$. This is due to McMullen. $SL(X,\omega)$ is a Fuchsian group, and it is never co-compact. $A \in SL(X,\omega)$ is hyperbolic if and only if it is pseudo-Anosov in affine coordinates.

Let $S = (X,\omega)$. In genus 2 on $H(2)$, we obtain a kind of Ratner's theorem for $SL_2\mathbb{R}$. Smillie shows that $SL(S)$ is a lattice if and only if the $SL_2\mathbb{R}$ orbit of S is closed. According to McMullen, this is true if and only if there is a hyperbolic element in $SL(S)$. According to Calta, this is equivalent to complete periodicity. Otherwise, $SL_2\mathbb{R}S$ is dense in $H^1(2)$. For $H(1,1)$, we have $SL(S)$ is a lattice if and only the $SL_2\mathbb{R}$ orbit of S is closed. It is possible that the $SL_2\mathbb{R}$ orbit of S is dense in $H(1,1)$. The last possibility is that the closure of the the $GL_2\mathbb{R}$ orbit of S is complex dimension 3 manifold that is linear in the period coordinates.

Theorem 2.3 (Hubert-Lanneau-Möller)**.** *In genus* 3*, there is an infinite family of examples where* S *is exceptional, the set of completely periodic directions is dense, and the closure of the* $SL_2\mathbb{R}$ *orbit of* S *is* $\mathcal{L}$*.*

$\mathcal{L}$ plays the role of a substratum. It is a sublocus of codimension 1 (the algebraic statement does not imply any ergodic properties.) $\mathcal{L} \subset H(2,2)$. $H(2,2)$ has two connected components: the hyperelliptic component, and the non-hyperelliptic component, which contains $\mathcal{L}$.

Theorem 2.4 (Hubert-Lanneau-Möller)**.** *In* $Q(1,1,1,1)$ *there exists and infinite family of quadratic differentials such that* (S,q) *is exceptional, the set of completely periodic directions is dense, and the closure of the* $SL_2\mathbb{R}$ *orbit of* (S,q) *is all of* $Q(1,1,1,1)$*.*

Theorem 2.5 (Hubert-Lanneau-Möller)**.** *There exist* $(X_n,\omega_n) \in \mathcal{L}(Q(1,1,1,1))$ *such that there exist hyperbolic elements in* $SL(X_n,\omega_n)$*, there is a dense set of completely periodic directions, and* $GL_2\mathbb{R}(X_n,\omega_n) = \mathcal{L}$*.*

Theorem 2.6. *If* $(X,\omega) \in \mathcal{L}$ *is obtained by the Thurston-Veech (Bouillabaise) construction, then there are two transverse parabolic elements in* $SL(X,\omega)$*, a direction which is completely periodic with cylinders of non-commensurable moduli, and if*

$$T_h = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

and

$$T_v = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

$[\mathbb{Q}(ab) : \mathbb{Q}] = 3$*, where* $\mathbb{Q}(ab)$ *denotes the trace field of* T_hT_v*.*