BILLIARDS IN GENUS TWO: VOLUMES AND COUNTING PROBLEMS

TALK GIVEN BY MATTHEW BAINBRIDGE

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1. Motivation

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g. We study $\Omega\mathcal{M}_g$ (holomorphic one-forms) on $\mathcal{M}_g$, and $\Omega_1\mathcal{M}_g \subset \Omega\mathcal{M}_g$ of unit area forms. We have a natural $GL_2^+\mathbb{R}$ action on $\Omega\mathcal{M}_g$.

Problem 1.1. *Classify the ergodic measures, looking at $SL_2\mathbb{R}$ acting on $\Omega_1\mathcal{M}_g$.*

Problem 1.2. *Compute volumes. This has been done in genus 2 by McMullen and Calta, and student by Eskin-Okounkov and Eskin-Masur-Schmull.*

The general setup is a s follows: We have a Riemann surface X with $0 \neq \omega$ a holomorphic one form. We get a translation structure as follows: for each sufficiently small open set U_α around p,

$$\varphi_\alpha(z) = \int_p^z \omega$$

defines an atlas on $X \setminus Z(\omega)$, and the transition functions are translations. This gives a flat cone metric, where the cone points are the zeros of the form. We also get a horizontal foliation.

If $A \in GL_2^+\mathbb{R}$. $\{A \circ \varphi_\alpha : U_\alpha \to \mathbb{C}\}$ gives a new atlas. $Y := X$ with pullback $\mathbb{C}$-structure, η the pullback of dz allows us to write $A \cdot (X, \omega) = (Y, \eta)$.

Denote by $\overline{n}$ the multi-index $(n_1, \dots, n_r)$. We let $\Omega\mathcal{M}_g(\overline{n})$ denote the forms with zeros of those orders. This is a suborbifold of the moduli space, an example of a stratum.

Theorem 1.3 (Veech)**.** *$\Omega\mathcal{M}_g(\overline{n})$ has an affine, i.e. $(GL_N\mathbb{Z}, \mathbb{C}^N \setminus \{0\})$-structure.*

2. Period coordinates

Let $(X_0, \omega_0) \in U \subset \Omega\mathcal{M}_g(\overline{n})$ for U simply connected. We take $(X, \omega) \mapsto [\omega] \in H^1(X, Z(\omega), \mathbb{C})$. We have $H^1(X_0, Z(\omega), \mathbb{C}) \cong \mathbb{C}^N$ via period coordinates. We get a measure μ on $\Omega\mathcal{M}_g(\overline{n})$ that is $SL_2\mathbb{R}$ invariant. We take

$$\mu_1 = \pi_*[\mu|_{\Omega_{\leq 1}}\mathcal{M}_g(\overline{n})],$$

where $\pi : \Omega\mathcal{M}_g(\overline{n}) \to \Omega_1\mathcal{M}_g(\overline{n})$.

Theorem 2.1 (Veech-Masur)**.** *μ_1 is finite, ergodic, $SL_2\mathbb{R}$-invariant.*

We observe that

$$vol(\mu_1) = \lim_{d\to\infty} \frac{S(d)}{d^N}$$

where $S(d) = \{$the number of square-tiled surfaces with $\leq d$ squares in $\Omega\mathcal{M}_g(\overline{n})\}$. This is all computed in Eskin-Okounkov.

3. Real multiplication

Let $D \in \mathbb{N}_{>0}$, $D \equiv 0, 1 \pmod 4$.

$$O_D = \mathbb{Z}[T]/(T^2 + bT + C),$$

where $b^2 - 4c = D$. This is the ring of integers in $\mathbb{Q}(\sqrt{D})$. Let A be an abelian surface, so we look at $\mathbb{C}^2/\Lambda$ where Λ is a lattice. We say A has real multiplication by O_D if there exists an embedding of rings $\rho : O_D \to End(A)$. For X a genus 2 Riemann surface, $Jac(X) = \Omega(X)^*/H_1(X,\mathbb{Z})$. We say $(X,\omega) \in \Omega\mathcal{M}_2$ is an eigenform for the real multiplication by O_D if $Jac(X)$ has real multiplication by O_D, where we have O_D acting on $\Omega(X) = \Omega^1(X) \oplus \Omega^2(X)$, and $\omega \in \Omega^i(X)$.

We look at ΩE_D, the locus of eigenforms for O_D.

Theorem 3.1 (McMullen, Calta)**.** *ΩE_D is $GL_2^+\mathbb{R}$-invariant.*

We look at period coordinates. Let $U \subset \Omega E_D(1,1)$. We get maps $U \to H_1(X, Z(\omega), \mathbb{C})$ and to $H^1_{O_D}(X, Z(\omega), \mathbb{C})$, where the latter is the O_D invariant subspace. We let μ_D be the pullback of Lebesgue measure. We use the same trick as before to produce a finite measure μ_D^1 on $\Omega_1 E_D(1,1)$. It is ergodic according to McMullen.

Theorem 3.2.

$$vol(\mu_D^1) = 8\pi\zeta_{\mathbb{Q}(\sqrt{D})}(-1) = \frac{8}{7}\pi\chi(\Omega E_D) = 4\pi\chi(\mathbb{H}\times\mathbb{H}/SL_2 O_D),$$

where this last space is called the Hilbert Modular Surface.

4. Relation to counting

Let $N((X,\omega), L) = \#\{$cylinders of length $\leq L$ on $(X,\omega)\}$.

Theorem 4.1. *For any $(X,\omega) \in \Omega E_D$, suppose $(X,\omega) \notin GL_2^+\mathbb{R}(decagon)$. Then, we get*

$$N((X,\omega), L) \sim \frac{15}{\pi}\frac{L^2}{Area(X,\omega)}.$$

What goes into the proof? First, results of Veech and Eskin-Masur that show that this function behaves like cL^2 where

$$c = \frac{\int_{\Omega\mathcal{M}_g} N((X,\omega), L)\, d\mu}{vol(\mu)\cdot L^2}.$$

Calta-Wortmann have classified all horocycle-invariant measures on ΩE_D.

The idea here is to consider $X_D = \mathbb{H}\times\mathbb{H}/SL_2 O_D$, which is viewed as the moduli space for abelian surfaces with real multiplication. We project and study the measure μ_D^1. We produce a foliation $\mathcal{F}_D$ on X_D by hyperbolic Riemann surfaces, which has a transverse invariant measure ω_1, a 2-form on X_D covered by $dx_1 dy_1/(2\pi y_1^2$. It turns out that $\omega_1 \times \mathcal{F}_D = \pi_*(\mu_D^1)$, and $[\omega_1]\cdot[\mathcal{F}_D] = vol(\pi_*\mu_D^1)$.