GEOMETRY OF THE THIN PART OF THE MODULI SPACE: TEICHMÜLLER GEODESICS AND RANDOM WALKS

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1. Counting closed geodesics.

This talk is based on work joint with Alex Eskin. We look at the dynamics of the geodesic flow on $Q^1\mathcal{M}_g$. We have a natural problem to count the number of closed geodesics of length $\leq L$, and we also look at the behavior of closed geodesics living in the thin part of $Q^1\mathcal{M}_g$, and also various strata in $\Omega^1\mathcal{M}_g$, along with $SL_2\mathbb{R}$ invariant subspaces.

We do the counting using Margulis' thesis. We also exploit the mixing of the geodesic flow and the nice properties of stable and unstable PL manifolds. Mixing allows us to estimate

$$\lim_{t\to\infty} \mu(g^t U \cap V) \to \mu(U)\mu(V).$$

We also look at the the natural map $Q^1\mathcal{M}_g \to \mathcal{M}_g$.

Theorem 1.1. *Let $h = 6g - 6$. Given $\epsilon > 0$ there exists a compact $K \subset \mathcal{M}_g$ such that the number of closed geodesics of length $\leq R$ not meeting $K = O(e^{(h+\epsilon-1)R})$.*

Remark 1.2. *There are examples by Hammenstadt-Rafi: closed geodesics living in $Q^1\mathcal{M}_g\backslash K$. We get a μ on Teichmüller space satisfying $vol(B_R(x)) \sim c(x)e^{hR}$ (Athreya-Bufetov-Eskin).*

2. Minsky's product-region theorem

The idea here is (Eskin-Margulis-(Mozes)/Athreya's thesis/Eskin-Masur).

Look at $T_g = T$ and fix a net $\{p_i\}$ with $d(p_i, p_j) \geq C$. For $X \in T$, suppose there is a p_i with $d(x, p_i) = O(1)$. Let $\lambda(0) = X$, and fix $\tau >> 1$. Take a random walk $\lambda(0), \lambda(1), \ldots)$ satisfying $d(\lambda(i), \lambda(i+1)) \leq \tau$. Estimating the number of random walks is unreasonable.

We look at extremal lengths. Let α be given. For α short, we have $Ext_\alpha \sim l_\alpha(X)$.

Theorem 2.1 (Kerckhoff's formula)**.**

$$d_T(x, y) = \sup_\alpha \log(\frac{\sqrt{Ext_\alpha(x)}}{\sqrt{Ext_\alpha(y)}})$$

T_ϵ is the locus where we get short curves. $T_\epsilon \to \mathbb{H}_1 \times \cdots \mathbb{H}_k \times T(S \setminus \gamma)$ is defined by Fenchel-Nielsen coordinates.

Remark 2.2. *For all $x \in \gamma$ a closed geodesic in $\mathcal{M}$ or length R, then the length of the shortest curve on x is $\geq e^{-(3g-3)R}$. If $d(p_i, p_j) \geq C$, the number of net points in $C(R)$ is a polynomial in R.*

There are various problems associated with these constructions. Let $P(x, R) = \{\{\lambda(i)\}^n, N = R/\tau\}$. There might be many closed geodesics corresponding to each path in $P(x, R)$. Also, $|P(x, R)| \leq e^{(h+\epsilon)R}$.