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Crystal Bases - what it is all about?

Outline:
- Representations of sl_2
- Representations of $U_q(sl_2)$
- Axiomatic Definitions of Crystals
- Affine crystals

① $\underline{sl_2}$: $\mathfrak{g} = sl_2(\mathbb{C})$ 3-dim. simple Lie Algebra generated by

$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and relations,

$$[h,e] = 2e \quad , \quad [h,f] = -2f \quad , \quad [e,f] = h \qquad (*)$$

$U(sl_2)$ associative algebra over $\mathbb{C}$ with 1 generated by e, f, h and relations $(*)$

linear functional : $\alpha \in (\mathbb{C}h)^*$ s.t. $\alpha(h) = 2$
then you can decompose :

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h,x] = \alpha(h)x\} = \mathbb{C}e$$

$$\mathfrak{g}_{-\alpha} = \{x \in \mathfrak{g} \mid [h,x] = -\alpha(h)x\} = \mathbb{C}f$$

$$\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [h,x] = 0\} = \mathbb{C}h$$

Root space decomposition $\mathfrak{g} = sl_2(\mathbb{C}) = \mathbb{C}f \oplus \mathbb{C}e \oplus \mathbb{C}h$

Definition: A vector space V is an sl_2-module if there is a bilinear map $sl_2 \times V \to V$ s.t. $(x, v) \mapsto x \cdot v$

$$[x, y].v = x(yv) - y(xv) \quad \forall x, y \in sl_2, \ v \in V.$$

Eigenspace decomposition:

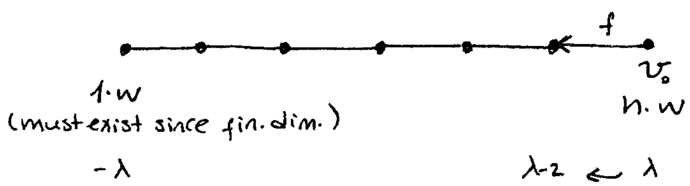
V - finite dimensional $sl_2(\mathbb{C})$-module.

$$V = \bigoplus_{\lambda} V_\lambda \quad \text{where} \quad V_\lambda = \{ v \in V \mid hv = \lambda v \}$$

V- finite dimensional module $\Rightarrow \exists \ v_0 \in V_\lambda$ for some λ s.t. $e v_0 = 0$ (and this will be the highest weight element)

Set $v_j = \frac{1}{j!} f^j v_0$ then we can check

$$h v_j = (\lambda - 2j) v_j$$
$$h v_j = (j+1) v_{j+1}$$
$$e v_j = (\lambda - j + 1) v_{j-1}$$



If you have n vectors then $\lambda = n$ in $h v_j = (\lambda - 2j) v_j$

② $U_q(sl_2)$

Now we can talk about $U_q(sl2)$ generators $F, E, H^{\pm 1}$ with relations

$$HE = q^2 EH$$
$$HF = q^{-2} FH \qquad (**)$$
$$EF - FE = \frac{H - H^{-1}}{q - q^{-1}}$$

We are working over $\mathbb{C}(q)$. We can now define $U_q(sl_2)$ to be the associative algebra with 1 generated by H, E, F with relations (**).

- All the previous things works similar for this case also

$l+1$ dim. irreps of $U_q(sl_2)$: $\quad H^{\pm 1} v_i = q^{\pm(\lambda - 2i)} v_i$

$$F v_i = \begin{cases} [i+1]_q v_{i+1}, & \text{if } i < l \\ 0, & \text{otherwise} \end{cases}$$

$$\left([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n} \right)$$

$$E v_i = \begin{cases} [\lambda - i + 1]_q v_{i-1}, & \text{if } i > 0 \\ 0, & \text{otherwise} \end{cases}$$

We start with v_0:

$$\begin{tikzcd} v_0 \arrow[d, "{[1]_q}"'] \\ F v_0 \arrow[d, "{[2]_q}"'] \\ F^2 v_0 \\ \vdots \arrow[d, "{[l]_q}"'] \\ F^l v_0 \end{tikzcd}$$

very similar to previous case.

we want to look at the "limit" as $q \to 0$.

Tensor Products:

Ex: V_1, 2-dim rep. of $U_q(sl_2)$

$$\begin{tikzcd} v_0 := v_+ \arrow[d] \\ F v_0 := v_- \end{tikzcd}$$

(two reps)

$$V_1 \otimes V_1 \cong V_2 \oplus V_0$$

$E v_+ = 0 \qquad F v_+ = v_- \qquad H v_+ = q v_+$

$E v_- = v_+ \qquad F v_- = 0 \qquad H v_- = q^{-1} v_-$

V_2 will be

$$v_+ \otimes v_+$$
$$v_+ \otimes v_- + q v_- \otimes v_+ \qquad \text{basis of } V_2$$
$$v_- \otimes v_-$$
$$v_- \otimes v_+ - q v_+ \otimes v_- \qquad \text{basis of } V_0$$

when we take the limit as $q \to 0$ the terms with q disappears. We can do similar things for any Lie Algebra.

Suggested Reading: Hong, Kang "Quantum Groups & Crystal Bases".

③ Axiomatic Definition of Crystals

$\mathfrak{g}$: Lie Algebra OR affine Kac-Moody Algebra

I : Index set

P : weight lattice

Definition: A $U_q(\mathfrak{g})$-crystal is a set B together with maps

weight $\leftarrow$ $wt : B \to P$

$e_i, f_i : B \to B \cup \{0\}$ $\quad i \in I$ $\quad$ such that

(1) $f_i b = b' \iff e_i b' = b$ for $b, b' \in B$

(2) $wt(e_i b) = wt(b) + \alpha_i \to$ simple root

$wt(f_i b) = wt(b) - \alpha_i$

Compatibility Condition (3) $\langle h_i, wt(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$

$$\underbrace{\bullet \xrightarrow{i} \bullet \xrightarrow{i}}_{\varepsilon_i(b)} b \underbrace{\xrightarrow{i} \bullet \xrightarrow{i} \bullet \xrightarrow{i} \bullet \xrightarrow{i} \bullet}_{\varphi_i(b)} \qquad b \xrightarrow{i} b' \qquad b' = f_i b$$

Some Examples:

Type $A_1 = sl_2$ $\qquad \alpha = \varepsilon_1 - \varepsilon_2$

$B(\square)$ $\qquad \boxed{1} \xrightarrow{1} \boxed{2}$ $\qquad$ Young

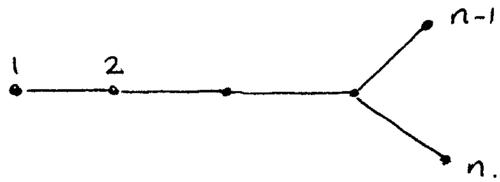
$B(\square\square)$ $\qquad \boxed{1|1} \xrightarrow{1} \boxed{1|2} \xrightarrow{1} \boxed{2|2}$ $\qquad$ Diagram.

$\vdots$

Type A_n, $\alpha = \varepsilon_i - \varepsilon_{i+1}$

$B(\square)$ $\qquad \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n} \boxed{n+1}$

Type D_n:
(crystal of type B)



$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

$\alpha_n = \varepsilon_{n-1} + \varepsilon_n$

$B(\square)$:

$$\begin{tikzcd} & & & \boxed{n} \arrow[dr, "n"] & & & \\ \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[u, "n-1"] \arrow[d, "n"'] & \boxed{\overline{n-1}} \arrow[r] & \cdots \arrow[r, "1"] & \boxed{\overline{1}} \\ & & & \boxed{\overline{n}} \arrow[ur, "n-1"'] & & & \end{tikzcd}$$

Tensor Product

B_1, B_2 $U_q(\mathfrak{g})$-crystals, can define $B_1 \otimes B_2$ as a set $B_1 \times B_2$ with $b_1 \otimes b_2 \in B_1 \otimes B_2$.

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2)$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b) \geq \varphi_i(b_2) \\ b_1 \otimes f_i b_2, & \text{else} \end{cases}$$

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2) \\ b_1 \otimes e_i b_2, & \text{else} \end{cases}$$

$B(\square) \otimes B'(\square)$

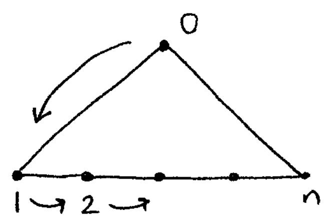
$$\begin{tikzcd}
1 \otimes 1 \arrow[d] & \\
1 \otimes 2 \arrow[d] & 2 \otimes 1 \arrow[ul, bend right] \\
2 \otimes 2 \arrow[ur, bend right] &
\end{tikzcd}$$

$2 \otimes 1 \rightsquigarrow$ the highest weight element.

④ Affine Crystals

Lie Algebras $\rightarrow$ affine Kac-Moody Lie Algebras (infinite dim. Lie Algebras).

EX: Type $A_n^{(1)}$



Rotate Dynkin Diagram and get affine crystals.