Modular Representations of Algebraic Groups: or to Characteristic Zero and Back Again, with Applications to Representations of Finite Groups of Lie Type in the Defining Characteristic

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MSRI Connections for Women
January 16–18, 2008
Outline

1 Characteristic Zero Lie Theory
   - Complex S.s. Lie Algebras and Their Irreducible Modules
   - Character and Dimension Formulae

2 Algebraic Groups in Positive Characteristic
   - A Few Basics
   - Chevalley Groups
   - Frobenius Morphisms
   - Representations of Algebraic Groups

3 Lusztig Conjecture
   - Quantum Enveloping Algebras
   - Representations and Lusztig’s Conjecture
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Set-up

- $\mathfrak{g}_\mathbb{C}$ complex s.s. Lie algebra, with Cartan subalgebra $\mathfrak{h}$, root system $\Phi \subset \mathfrak{h}_\mathbb{C}^*$, with Weyl group $W$ and base of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$

- $\varpi_i$ the fundamental dominant (integral) weight corresponding to simple root $\alpha_i$, weight lattice $\Lambda$ with partial order $\leq$, root lattice $\Lambda_r$, dominant (integral) weights $\lambda^+, \rho := \frac{1}{2} \alpha \in \Phi^+ = \sum_{i=1}^\ell \varpi_i$

- Universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$; can be defined as associative algebra (w/1) on generators $e_{\alpha_i}, f_{\alpha_i}, \alpha_i \in \Pi, h_j, j = 1, \ldots, \dim(\Lambda)$, satisfying the Serre relations
- Details

- Associated to base $\Pi$, triangular decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{n}_- \oplus \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_+ \cong \mathfrak{h}_\mathbb{C} \oplus \mathfrak{b}_\mathbb{C}^+$ (for $\mathfrak{b}_\mathbb{C}^+ := \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_+; \mathfrak{n}_- := \mathfrak{h}_\mathbb{C} \oplus \mathfrak{n}_-$)
  with corresponding triangular decomposition (v. space isos.)
  $U(\mathfrak{g}_\mathbb{C}) \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{h}_\mathbb{C}) \otimes_{\mathbb{C}} U(\mathfrak{n}_+) \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{b}_\mathbb{C}^+)$
Irreducible $g$-modules

- Given $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, have Verma (or standard) module $V(\lambda)$, a $g_{\mathbb{C}}$-module of highest weight $\lambda$.
- $V(\lambda) = U(g_{\mathbb{C}}) \otimes U(b^+_{\mathbb{C}}) \mathbb{C}_\lambda$, for $\mathbb{C}_\lambda$ 1-dl. $b^+$ rep. w/basis $v_\lambda$ satisfying $n^+.v_\lambda = 0$, and $h.v_\lambda = h(\lambda)v_\lambda \forall h \in \mathfrak{h}_{\mathbb{C}}$.
- $V(\lambda) = U(n^-_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as v. spaces, so is infinite-dimensional.
- $V(\lambda)$ has weight space decomposition $V(\lambda) = \bigoplus_{\mu \in \mathfrak{h}_{\mathbb{C}}^*} V(\lambda)_\mu$, $\mu \leq \lambda$. Although there are infinitely many weights, $\dim(V(\lambda)_\mu) < \infty \ \forall \mu$.
- Every $g_{\mathbb{C}}$-module of highest weight $\lambda$ is a homomorphic image of $V(\lambda)$.
- $V(\lambda)$ has a unique maximal submodule and irreducible head $L(\lambda)_{\mathbb{C}}$; $\dim((L(\lambda)_{\mathbb{C}})_\lambda) = 1$.
- $L(\lambda)_{\mathbb{C}}$ is finite dimensional $\iff \lambda \in \Lambda^+$.
- Consequently, the finite dimensional irreducible $g_{\mathbb{C}}$-modules are parameterized (up to isomorphism) by their highest weights, and $\{L(\lambda)_{\mathbb{C}} \mid \lambda \in \Lambda^+\}$ is a representative list of all irreducible finite-dimensional $g_{\mathbb{C}}$-modules.
Weight Structures of Verma Modules and Irreducibles, $\lambda \in \Lambda^+$

In terms of weight spaces,

$$
V(\lambda) = \begin{array}{c}
V(\lambda)_{\lambda} \\
V(\lambda)_{\mu} \\
\vdots \\
V(\lambda)_{\nu} \\
\vdots \\
\end{array}, \quad \lambda > \mu, \ldots, \nu, \ldots \Rightarrow \begin{array}{c}
L(\lambda)_{\lambda} \\
L(\lambda)_{\eta} \\
\vdots \\
L(\lambda)_{\zeta} \\
\end{array}, \quad \lambda > \eta, \ldots, \zeta.
$$
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Characters, Weyl Modules, and Irreducible Modules

For any $\mathfrak{g}_C$-module $V$ which is a direct sum $V = \bigoplus_{\mu \in \mathfrak{h}^*} \text{of fin. dl.}$ $\mathfrak{h}_C$-weight spaces, one has the character $\text{ch}(V) : \mathfrak{h}_C^* \to \mathbb{Z}$, $\text{ch}(V)(\mu) = \dim(V_\mu)$ (& formal character $\text{ch}(V) = \sum_{\mu \in \mathfrak{h}_C^*} (\dim(V_\mu) e^\mu)$)

For example, $V(\lambda) = \mathcal{U}(\mathfrak{n}^-) \otimes C \lambda$ as v. spaces $\Rightarrow$ $\dim(V(\lambda)_{\mu}) = \#$ ways to write $\mu$ as $\lambda - \sum_{\alpha_j \in \Phi^+} m_i \alpha_j$ ($\lambda$ a non-neg. sum integral sum of pos. roots); consequently, can show $\text{ch}(V(\lambda)) = e^\lambda / \prod_{\alpha > 0}(1 - e^{-\alpha}) = e^{\lambda + \rho} / (\prod_{\alpha > 0}(e^{\alpha/2} - e^{-\alpha/2}))$.

From $\dim(L(\lambda)) < \infty$, $\dim((L(\lambda)_C)_{\lambda}) = 1$, and weight structure, can show the $\text{ch}(L(\lambda)_C)$, $\lambda \in \Lambda^+$ form a basis for $\mathbb{Z}[\Lambda]^W$, where $W$ acts via $w.e^\mu = e^{w\mu}$.

If $\dim(V) < \infty$, then $\text{ch}(V) \in \mathbb{Z}[\Lambda]^W$, so is determined by $\text{ch}(L(\lambda)_C)$, $\lambda \in \Lambda^+$. (Here, moreover, Weyl's Complete Reducibility Thm. $\Rightarrow$ $V$ is a direct sum of some $L(\lambda)_C$s, $\lambda \in \Lambda^+$, so certainly all finite dimensional $V$ are determined by knowledge of $\text{ch}(V)$.)
Characters, Weyl Modules, and Irreducible Modules

For any $g_{\mathbb{C}}$-module $V$ which is a direct sum $V = \bigoplus_{\mu \in \mathfrak{h}^*} \text{of fin. dl. }$ $\mathfrak{h}_{\mathbb{C}}$-weight spaces, one has the character $\text{ch}(V) : \mathfrak{h}_{\mathbb{C}}^* \to \mathbb{Z}$, $\text{ch}(V)(\mu) = \dim(V_\mu)$ (\& formal character $\text{ch}(V) = \sum_{\mu \in \mathfrak{h}_{\mathbb{C}}^*} (\dim(V_\mu)e^\mu)$)

For example, $V(\lambda) = U(n^-) \otimes_{\mathbb{C}} \mathbb{C}\lambda$ as v. spaces $\Rightarrow$ $\dim(V(\lambda)_\mu) = \# \text{ ways to write } \mu$ as $\lambda - \sum_{\alpha_i \in \Phi^+} m_i\alpha_i \text{ (}\lambda-\text{a non-neg. sum integral sum of pos. roots)}$; consequently, can show $\text{ch}(V(\lambda)) = e^\lambda/\Pi_{\alpha>0}(1 - e^{-\alpha}) = e^{\lambda+\rho}/(\Pi_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2}))$.

From $\dim(L(\lambda)) < \infty$, $\dim((L(\lambda)_{\mathbb{C}})_\lambda) = 1$, and weight structure, can show the $\text{ch}(L(\lambda)_{\mathbb{C}}), \lambda \in \Lambda^+$ form a basis for $\mathbb{Z}[(\Lambda)]^W$, where $W$ acts via $w.e^\mu = e^{w\mu}$.

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Characters, Weyl Modules, and Irreducible Modules

For any $g_C$-module $V$ which is a direct sum $V = \bigoplus_{\mu \in \mathfrak{h}^*} \text{fin. dl.}$ $\mathfrak{h}_C$-weight spaces, one has the character $\chi(V) : \mathfrak{h}_C^* \to \mathbb{Z}$, $\chi(V)(\mu) = \dim(V_\mu)$ (& formal character $\chi(V) = \sum_{\mu \in \mathfrak{h}_C^*} (\dim(V_\mu) e^\mu)$)

For example, $V(\lambda) = \mathcal{U}(n^-) \otimes \mathbb{C}_\lambda$ as v. spaces $\Rightarrow$ $\dim(V(\lambda)_\mu) = \# \text{ ways to write } \mu$ as $\lambda - \sum_{\alpha_i \in \Phi^+} m_i \alpha_i$ ($\lambda$ — a non-neg. sum integral sum of pos. roots); consequently, can show $\chi(V(\lambda)) = e^{\lambda} / \prod_{\alpha > 0} (1 - e^{-\alpha}) = e^{\lambda + \rho} / (\prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2}))$.

From $\dim(L(\lambda)) < \infty$, $\dim((L(\lambda)_C)_\lambda) = 1$, and weight structure, can show the $\chi(L(\lambda)_C)$, $\lambda \in \Lambda^+$ form a basis for $\mathbb{Z}[(\Lambda)]^W$, where $W$ acts via $w . e^\mu = e^{w \mu}$.

If $\dim(V) < \infty$, then $\chi(V) \in \mathbb{Z}[(\Lambda)]^W$, so is determined by $\chi(L(\lambda)_C)$, $\lambda \in \Lambda^+$. (Here, moreover, Weyl’s Complete Reducibility Thm. $\Rightarrow V = \text{a direct sum of some } L(\lambda)_C$s, $\lambda \in \Lambda^+$, so certainly all finite dimensional $V$ are determined by knowledge of $\chi(V)$.)

Characters, Weyl Modules, and Irreducible Modules

For any $g_{\mathbb{C}}$-module $V$ which is a direct sum $V = \bigoplus_{\mu \in \mathfrak{h}^*} \text{of fin. dl.}$ $\mathfrak{h}_{\mathbb{C}}$-weight spaces, one has the character $\text{ch}(V) : \mathfrak{h}_{\mathbb{C}}^* \rightarrow \mathbb{Z}$, $\text{ch}(V)(\mu) = \dim(V_\mu)$ (& formal character $\text{ch}(V) = \sum_{\mu \in \mathfrak{h}_{\mathbb{C}}^*} (\dim(V_\mu) e^\mu)$)

For example, $V(\lambda) = U(n^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as v. spaces $\Rightarrow$ $\dim(V(\lambda)_\mu) = \# \text{ ways to write } \mu \text{ as } \lambda - \sum_{\alpha_i \in \Phi^+} m_i \alpha_i \text{ (} \lambda \text{ a non-neg. sum integral sum of pos. roots)}$; consequently, can show $\text{ch}(V(\lambda)) = e^\lambda / \prod_{\alpha > 0} (1 - e^{-\alpha}) = e^{\lambda + \rho} / (\prod_{\alpha > 0} (e^{\alpha/2} - e^{-\alpha/2}))$.

From $\dim(L(\lambda)) < \infty$, $\dim((L(\lambda)_\mathbb{C})_\lambda) = 1$, and weight structure, can show the $\text{ch}(L(\lambda)_\mathbb{C}), \lambda \in \Lambda^+$ form a basis for $\mathbb{Z}[(\Lambda)]^W$, where $W$ acts via $w.e^\mu = e^{w^\mu}$.

If $\dim(V) < \infty$, then $\text{ch}(V) \in \mathbb{Z}[(\Lambda)]^W$, so is determined by $\text{ch}(L(\lambda)_\mathbb{C}), \lambda \in \Lambda^+$. (Here, moreover, Weyl’s Complete Reducibility Thm. $\Rightarrow$ $V = \text{a direct sum of some } L(\lambda)_\mathbb{C} \text{s, } \lambda \in \Lambda^+$, so certainly all finite dimensional $V$ are determined by knowledge of $\text{ch}(V)$.)
Characters, Weyl modules, and Irreducible Modules

Let $\lambda \in \Lambda^+$. 

- **KEY FACT:** $V(\lambda)$ has finite comp. series w/factors $L(\mu)$, and multiplicity $[V(\lambda) : L(\mu)] \neq 0 \iff \mu = w \cdot \lambda \quad \exists w \in W$. Necessarily, $\mu < \lambda$; also recall $[V(\lambda)_{\mathbb{C}} : L(\lambda)_{\mathbb{C}}] = 1$.
- Thus $\text{ch}(V(\lambda)) = \sum_{w \in W} a_w \text{ch}(L(w \cdot \lambda)_{\mathbb{C}})$, $a_w \in \mathbb{Z}_{\geq 0}$, $a_1 = 1$.
- Likewise, for $w \cdot \lambda \leq \lambda$, $\text{ch}(V(w \cdot \lambda)) = \sum_{y \in W} a_y \text{ch}(L(y \cdot \lambda)_{\mathbb{C}})$, $y \cdot \lambda \leq w \cdot \lambda$.

\[ V(\lambda) = \begin{array}{c} L(\lambda) \\
L(\mu) \\
\vdots \\
L(w \cdot \lambda) \\
\vdots \\
L(\nu) \end{array} \]

$\mu, \ldots, w \cdot \lambda, \ldots, \nu < \lambda, w \in W, w \cdot \lambda := w(\lambda + \rho) - \rho.$
From system of equations for the $\text{ch}(V(w \cdot \lambda))$ in terms of the $\text{ch}(L(y \cdot \lambda)_\mathbb{C})$, can order $\{w \cdot \lambda \mid w \cdot \lambda \leq \lambda\}$ to get square upper triangular $\mathbb{Z}$-matrix $A$ with $\text{diag}(A) = (1, 1, \ldots, 1)$. Then $A^{-1}$ produces an equation

$$\text{ch}(L(\lambda)_\mathbb{C}) = \sum_{w \in W} b_w \text{ch}(V(w \cdot \lambda)) \quad \exists b_w \in \mathbb{Z}.$$  

Using formula for $\text{ch}(V(\mu))$ and examining Weyl group action on each side of equation above yields Weyl’s Character Formula

$$\text{ch} L(\lambda)_\mathbb{C} = \sum_{w \in W} (-1)^{\ell(w)} \text{ch}(V(w \cdot \lambda)).$$

The $\text{ch}(V(w \cdot \lambda))$ are known!

So $\text{ch}(L(\lambda)_\mathbb{C}), \lambda \in \Lambda^+$ are all known,

so $\text{ch}(V)$ for any finite dl. $g$-module $V$ is known.

From Weyl’s Character Formula, one also obtains $\dim(L(\lambda)_\mathbb{C}) \forall \lambda \in \Lambda^+$. 

Proof details
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A Few Basics about Affine Algebraic Groups

Set $k = \overline{\mathbb{F}}_p$.

- An affine (i.e., linear) algebraic group $G$ over $k$ can be viewed as an affine algebraic variety $G \subset k^n$ ($\exists n$) with a compatible group structure. Have Borel $B$, maximal torus $T$, characters $X(T)$, cocharacters $Y(T)$, dominant weights $X(T)^+$...

- Coordinate algebra $k[G]$ of such a variety is a finitely generated reduced commutative $k$-algebra. It is also a Hopf algebra.

- Can also consider $G$ functorially as a representable functor from category of commutative $k$-algebras to category of groups, with $G(A) = \text{Hom}_{k-\text{alg}}(k[G], A)$, so that $G(k)$ identifies with $G$ originally regarded as an affine algebraic variety.

- Can expand functorial perspective to use representing algebras $R \in k - \text{alg}$ in place of $k[G]$ which are fin. gen., but not necessarily reduced ("algebraic affine $k$-group schemes") or just commutative, but not even fin. gen. ("affine $k$-group scheme"). If $R$ is not just fin. generated, but f. dl., call $\text{Hom}_{k-\text{alg}}(R, -)$ finite; includes "infinitesimal group schemes".
A linear algebraic group $G$ is a closed subgroup of $GL_n(k)$ for some $n$. There is a natural notion of a (rational) $G$-module $V$, e.g., alg. group hom $G \to GL(V)$ (by assumption, $\dim(V) < \infty$). There is a compatible notion of $G$-modules for group schemes via group scheme maps $G \to GL(V)$, or comodules for the Hopf algebra $k[G]$.

Complete reducibility for (rational, f. dl.) $G$-modules does not hold. (Consider $SL_2(k)$ acting on symmetric powers $S^i(V)$, $i = p^r$, for natural module $V = k^2$ with standard basis $u, v$ as an example.)
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$g_C$ complex s.s. Lie algebra with Cartan subalgebra $h_C$, weight lattice $\Lambda$, root lattice $\Lambda_r$.

For faithful finite-dl. $g_C$ module $V$, $\Lambda_V$ sublattice of $\Lambda_r$ generated by all weights of $h_C$ on $V$, have $\Lambda_r \subset \Lambda_V \subset \Lambda$.

$\{X_\alpha \in g_C : \alpha \in \Phi\} \cup \{H_i \in h_C : \alpha_i \in \Pi\}$ Chevalley basis for $g_C$.

$U_Z$ Kostant $\mathbb{Z}$-form of enveloping algebra $U(g_C)$, subalgebra generated by all $X_\alpha^{(n)} := \frac{X_\alpha^n}{n!}$, $\alpha \in \Phi$, $n \in \mathbb{N}$.

\exists lattice $V_Z$ in $V$ invariant under $U_Z$; for $k = \overline{\mathbb{F}_p}$, set $V_k := V_Z \otimes_{\mathbb{Z}} k$.

For $t \in k$, $\alpha \in \Phi$, $\exp(tX_\alpha) : V_k \to V_k$ defined by $\exp(tX_\alpha)(v \otimes a) = \sum_{n=0}^{\infty} v \otimes t^n a$ is well-defined automorphism.

Set $G$ to be subgroup of $\text{Aut}(V_k)$ generated by all $\exp(tX_\alpha)$, $t \in k$, $\alpha \in \Phi$.

By def., $G$ is a Chevalley group; is s.s. alg. group defined over $\mathbb{F}_p$ w/ $g := \text{Lie}(G) = g_Z \otimes_{\mathbb{Z}} k$ for $g_Z$ the lattice in $g_C$ preserving the $\mathbb{Z}$-form $V_Z$.

Chevalley group $G$ has maximal torus $T$ with $X(T) = \Lambda_V$, root lattice $\Lambda_r$ and weight lattice $\Lambda$ w.r.t. $T$.

Chevalley group $G$ is universal if $\Lambda_V = \Lambda$ iff $G$ is simply connected.
Chevalley Groups, the Kostant $\mathcal{Z}$-form, and Hyperalgebras

$$\text{Dist}(G_C) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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Frobenius Morphisms and Finite Groups of Lie Type

- $k = \overline{\mathbb{F}_p}$
- $G$ connected affine algebraic group defined over $\mathbb{F}_p \subset k$ (so there is a Hopf algebra $A_0$ such that $k[G] \cong k \otimes_{\mathbb{F}_p} A_0$)
- $F : G \to G$ Frobenius morphism (induced by comorphism $F^* : k[G] \to k[G]$, $(\alpha \otimes f) \mapsto \alpha \otimes f^p$)
- $r^{th}$ Frobenius morphism $F^r$ ($= r^{th}$ power of $F$), $r \geq 1$
- For $q = p^r$, $G(\mathbb{F}_q) := G^{F^r} = \{g \in G | F^r(g) = g\}$, finite group of $\mathbb{F}_q$ rational points of $G$
- More generally, can consider ‘generalized Frobenius morphisms’ $F_{gen} : G \to G$, characterized by $F_{gen}^m = F^r$ for some $m, r \geq 1$
- Every finite group of Lie type arises as $G^{F_{gen}}$ for some generalized Frobenius morphism.
Frobenius Morphisms and Frobenius Kernels

- $k = \overline{\mathbb{F}_p}$, $G/k$ affine algebraic group defined over $\mathbb{F}_p$
- $F : G \to G$ Frobenius morphism (induced by comorphism $F^* : k[G] \to k[G]$, $(\alpha \otimes f) \mapsto \alpha \otimes f^p$)
- $r^{th}$ Frobenius morphism $F^r$ ($= r^{th}$ power of $F$), $r \geq 1$
- $G_r = \ker (F^r)$, $r^{th}$ Frobenius kernel, normal subgroup of $G$
- $G_r$ is an infinitesimal group scheme, a ‘nontrivial trivial group’: $G_r(K) = \text{Hom}_{k-\text{alg}}(k[G_r], K) = \{e\}$, the trivial group, for any field extension $K \supset k$. 
Chevalley Groups, the Kostant $\mathbb{Z}$-form, Hyperalgebras and Frobenius Kernels

$G_C = G(V_C, \mathbb{C}) \subset GL(V_C)$

$G = G(V_k, k) \subset GL(V_k)$
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Universal Chevalley group $G/k$, defined over $\mathbb{F}_p \subset k = \overline{\mathbb{F}}_p$

$F : G \to G$ Frobenius morphism (and $F_{gen} : G \to G$ generalized Frobenius morphism)

By Curtis and Steinberg, the irreducible representations of a finite group $G^{F_{gen}}$ of Lie type in \textit{defining characteristic} are ‘determined’ by the representation theory of the overarching algebraic group $G$.

By Steinberg’s Tensor Product Theorem, the irreducible representations for $G$ in positive characteristic $p$ are ‘determined’ by the representation theory of the Frobenius kernel $G_1 = \ker (F)$.

\begin{itemize}
  \item \textbf{GOAL:} Determine characters and dimensions of $L(\lambda), \lambda \in X_1(T)^+$.  
\end{itemize}
Comparisons: Reps. for $g_C$ and $G$

\[\{\text{Complex s.s. Lie algebra } g_C\} \iff \{\text{S.s. s. conn. algebraic group } G/k\} \]
\[(1)\]

Standard Modules

\[\{\text{Verma module } V(\lambda)\} \iff \{\text{Weyl module } \Delta(\lambda) := (L(\lambda)_C)_Z \otimes k\} \]
\[(2)\]

Costandard Modules

\[\{\text{Dual Verma module}\} \iff \{\text{Induced module } \nabla(\lambda) = H^0(\lambda)\} \]
\[(3)\]

Irreducible Finite-Dim’l Modules ($\lambda$ dominant)

\[\{L(\lambda)_C\} \iff \{L(\lambda) := \text{head}(\Delta(\lambda))\}\]
\[(4)\]
PROBLEM: $\Delta(\lambda)$ is not irreducible.

FIX: From weight structures, and Weyl’s Character Formula, still get picture below, so as before, get $\text{ch}(L(\lambda)) = \sum_{\mu} b_{\mu,\lambda} \text{ch} \Delta(\mu)$, $b_{\lambda,\lambda} = 1$. Then, solve for $b_{\mu,\lambda}$!

PROBLEM: Proof of Weyl’s Char. Formula (really, any proof) makes use of $[V(\lambda) : L(\mu)] \neq 0 \Rightarrow \mu = \omega \cdot \lambda$, not just $\mu < \lambda$.

Proof of Weyl’s Char. Formula

Linkage Principle: $[\Delta(\lambda), L(\mu)] \neq 0 \Rightarrow \mu \in W \cdot \lambda + p\mathbb{Z}\Phi$. 
PROBLEM: $\Delta(\lambda)$ is not irreducible.

FIX: From weight structures, and Weyl’s Character Formula, still get picture below, so as before, get $\text{ch}(L(\lambda)) = \sum_{\mu} b_{\mu,\lambda} \text{ch} \Delta(\mu)$, $b_{\lambda,\lambda} = 1$. Then, solve for $b_{\mu,\lambda}$!

PROBLEM: Proof of Weyl’s Char. Formula (really, any proof) makes use of $[V(\lambda) : L(\mu)_{\mathbb{C}}] \neq 0 \Rightarrow \mu = w \cdot \lambda$, not just $\mu < \lambda$. 

Proof of Weyl’s Char. Formula

Linkage Principle: $[\Delta(\lambda), L(\mu)] \neq 0 \Rightarrow \mu \in W \cdot \lambda + p\mathbb{Z}\Phi$. 
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Proof of Weyl’s Char. Formula

Linkage Principle: $[\Delta(\lambda), L(\mu)] \neq 0 \Rightarrow \mu \in W \cdot \lambda + p\mathbb{Z}\Phi$. 
Character Theories: \( g_C \) and \( G \)

\[
\begin{align*}
\{ \text{Complex s.s. Lie algebra } g_C \} & \quad \iff \quad \{ \text{universal Chevalley group } G/k, \text{ char } (k) = p > 0 \} \\
\end{align*}
\]

Characters

\[
\begin{align*}
\begin{cases}
\text{ch } L(\lambda)_C \\
\sum_{w \in W} b_w \text{ ch } V(w \cdot \lambda)
\end{cases}
& = 
\begin{cases}
\text{ch } L(\lambda) \\
\sum_{w \in W_p} b_w \text{ ch } \Delta(w \cdot \lambda)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\text{ch } L(\lambda)_C \text{ known since } \\
\text{ch } V(w \cdot \lambda) \text{ known, and } b_w = (-1)^{\ell(\lambda)} \\
\text{(Weyl’s Char. Form.)}
\end{cases}
& \iff 
\begin{cases}
\text{ch } L(\lambda) \text{ known} \\
\text{(via Weyl’s Char. Form.)}, \text{ but } b_w \text{ ‘known’ only up to Lusztig’s conjecture}
\end{cases}
\end{align*}
\]
Assume $G$ is a universal Chevalley group constructed from complex s.s. Lie algebra $g_C$.

- Lusztig’s conjecture, below, asserts that the coefficients $b_w$ are in effect given by the values at 1 of certain polynomials $P_{y,w}$, called Kazhdan-Lusztig polynomials, associated with the Coxeter group $W_p$.

- Though Lusztig’s formula is known to be correct for $p \gg h$, where $h = 1 + \langle \rho, \alpha_0^\vee \rangle$ is the Coxeter number of $\Phi$ ($\alpha_0$ is the longest short root of $\Phi$), a lower bound for $p$ is not known.

- Additional terminology: we say that a dominant weight $\mu$ lies in the Jantzen region if $\langle \mu + \rho, \alpha_0^\vee \rangle \leq p(p - h + 2)$. 
Assume $G$ is a universal Chevalley group constructed from complex s.s. Lie algebra $\mathfrak{g}_C$.

- Lusztig’s conjecture, below, asserts that the coefficients $b_w$ are in effect given by the values at 1 of certain polynomials $P_{y,w}$, called Kazhdan-Lusztig polynomials, associated with the Coxeter group $W_P$.

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Assume $G$ is a universal Chevalley group constructed from complex s.s. Lie algebra $\mathfrak{g}_C$.

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- Additional terminology: we say that a dominant weight $\mu$ lies in the Jantzen region if $\langle \mu + \rho, \alpha_0^\vee \rangle \leq p(p - h + 2)$. 
Conjecture (Lusztig, 1979)

Let $\lambda$ be a weight in the Jantzen region (which includes all restricted weights if $p \geq 2h - 2$, $h$ the Coxeter number of $\Phi$). Then if $p \geq h$, $\dim L(\lambda)_\nu$ is given as follows: Choose $w$ in the affine Weyl group $W_p = p\mathbb{Z}\Phi \rtimes W$ such that $\lambda = w \cdot \lambda_0$, for some $\lambda_0$ (unique) with $-p \leq (\lambda_0 + \rho)(H_\alpha) \leq 0$ for all $\alpha \in \Phi^+$. (We say that $\lambda_0$ is in the antidominant lowest alcove.) Let $w_0$ denote the longest element of $W$.

Then

$$\dim L(\lambda)_\gamma = \sum (-1)^{l(w) - l(y)} P_{y,w}(1) \dim \Delta(\omega_0 y \cdot \lambda_0)_\gamma$$

(8)

where the sum is taken over all $y \in W$ such that $w_0 y \cdot \lambda_0$ is dominant and $w_0 y \cdot \lambda_0 \leq w_0 w \cdot \lambda_0 = \lambda$, $\Delta(\omega_0 y \cdot \lambda_0)$ is the Weyl module of highest weight $w_0 y \cdot \lambda_0$, and $P_{y,w}$ is a Kazhdan-Lusztig polynomial associated with the Coxeter group $W_p$. 
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Set-up

For indeterminate $\nu$, $\mathbb{U}_\nu = \mathbb{U}_\nu(\mathcal{R})$ quantum (or quantized) enveloping algebra (QEA) (a.k.a. ‘quantum group’) associated to root datum $\mathcal{R} = (\Pi, X, \Pi^\vee, X^\vee)$ is a $\mathbb{Q}(\nu)$-algebra (w/1) defined by generators $E_i, F_i, i \leftrightarrow \alpha_i \in \Pi, K_h, h \in X^\vee$, subject to the quantum Serre relations involving $\nu$.

For $\mathcal{A} := \mathbb{Z}[\nu, \nu^{-1}]$, there is Lusztig integral form $\mathbb{U}_\mathcal{A}$ of $\mathbb{U}_\nu$, an $\mathcal{A}$-subalgebra generated by appropriately defined ‘divided powers’ $E_i^{(n)}, F_i^{(n)}, n \in \mathbb{Z} \geq 0$ along with certain expressions involving the $K_h$s (see e.g., [2, H.5]).

For any commutative ring $K$ and any invertible element $q \in K$, there are specializations $\mathbb{U}_{q,K} := \mathbb{U}_\mathcal{A} \otimes_\mathcal{A} K$ arising from the $\mathcal{A}$-module structure on $K$ given by the (unique) ring homomorphism $\epsilon_q : \mathcal{A} \rightarrow K$, $\nu \mapsto q$.

For each algebra above, there is an associated triangular decomposition, with $+$-part associated to $E_i$-type generators, $-$-part associated to $F_i$-type generators, and 0-part associated to appropriate $K_h$ expressions.
From here on, assume \( \mathcal{R} \) is root datum of a simply connected, semisimple algebraic group \( G/k \), defined over \( \overline{\mathbb{F}}_p \subset k = \overline{k} \). Also, let \( K := \mathbb{Q}(q), p^{th} \) cyclotomic field (so \( q \) is a prim. \( p^{th} \)-root of unity). Then

- \( \mathbb{U}_{q,K} \) is generated over \( K \) by all \( E_i, E_i^{(p)}, F_i, F_i^{(p)}, K_h^\pm 1 \)
- ‘Small quantum group’ \( \mathfrak{u}_{q,K} \subset \mathbb{U}_{q,K} \) is subalgebra generated by all \( E_i, F_i, K_h^\pm 1 \); has finite dimension \( \dim(\mathfrak{u}_{q,K}) = 2|\prod p^{\dim(G)}| \).

Compare w/Alg. Gps.
Outline

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Comparisons: QEAs\(^1\) and Algebraic Groups

\[
\left\{ \begin{array}{l}
\text{QEA} \quad U_{q,K}(\mathcal{R}) , \quad q \text{ \(\ell\)th root of unity in field } K , \\
\text{char} (K) = 0
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{Universal Chevalley group} \\
G/k \text{ of root datum } \mathcal{R} , \\
\text{char} (k) = p > 0
\end{array} \right\}
\] (9)

Standard Modules

\[
\left\{ \begin{array}{l}
\text{quantum Verma module } V_q(\lambda)
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{Weyl module } \Delta(\lambda)
\end{array} \right\}
\] (10)

Costandard Modules

\[
\left\{ \begin{array}{l}
\text{quantum induced module } H^0_q(\lambda)
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{Induced module } \nabla(\lambda) = H^0(\lambda)
\end{array} \right\}
\] (11)

Irreducible Finite-Dim’l Modules

\[
\left\{ L_q(\lambda) \right\} \leftrightarrow \left\{ L(\lambda) \right\}
\] (12)

\(^1\)WLOG, modules are integrable type 1
Character Theories: QEA and Algebraic Groups

\[
\begin{align*}
\begin{cases}
\text{QEA } U_{q,K}(\mathfrak{h}), \ q \ell\text{-th-root of unity in field } K, \ \text{char}(K) = 0 \\
\end{cases}
\end{align*}
\begin{align*}
\leftrightarrow
\begin{cases}
\text{universal Chevalley group } G/k, \ \text{root datum } \mathfrak{r}, \ \text{char}(k) = p > 0
\end{cases}
\end{align*}
\]

Characters

\[
\begin{align*}
\begin{cases}
\text{known (Lusztig's Conjecture (8) with } L_q(\lambda), \ \text{resp., } \\
\nabla q(\lambda) \text{ instead of } L(\lambda), \ \text{resp., } \nabla(\lambda)) \text{ is a theorem}^2
\end{cases}
\end{align*}
\begin{align*}
\leftrightarrow
\begin{cases}
\text{known (via Lusztig's Conj. for QEA's), } \\
\text{but w/out precise lower bound on } p
\end{cases}
\end{align*}
\]

\[^2\text{With a few limitations.}\]
Some Steps in Proof of LC for Algebraic Groups

Reduction from algebraic to quantum group case: Assume \( p \geq 2h - 2 \).

\[
\text{ch } L(\mu) = \text{ch } L_q(\mu)
\]

\[
\Leftrightarrow \text{ch } \tilde{L}_1(\mu) = \text{ch } L_q(\tilde{\mu})
\]

for LHS = simple \( G_1 T \)-module, RHS = simple \( u_{q,k} U_{q,K}^0 \)-module “of type 1” (i.e., \( K^p_h \) acts as 1)

\[
\Leftrightarrow [\tilde{Z}_1(\mu) : \tilde{L}_1(\nu)] = [\tilde{Z}_q(\mu) : \tilde{L}_q(\nu)] \forall \mu, \nu \in X(T),
\]

for \( \tilde{Z}_1(\mu) \) induction from \( B_1 T \) to \( G_1 T \) and \( \tilde{Z}_q(\mu) \) induction from \( u_{q,k}^{-} U_{q,K}^0 \) to \( u_{q,k}^{-} U_{q,K}^0 \)

\[
\Leftrightarrow \text{ch } \tilde{Q}_1(\lambda) = \text{ch } \tilde{Q}_q(\lambda) \forall \lambda \in X(T),
\]

for LHS module = injective hull of \( \tilde{L}_1(\lambda) \) in category of \( G_1 T \)-modules, and RHS module = injective hull in category of \( u_{q,k}^{-} U_{q,K}^0 \)-modules. Last equality holds iff holds for all \( \lambda \in X(T)^+ \) with \( < \lambda + \rho, \alpha^\vee > < p \) for all \( \alpha \in \Pi \).

[AJS] \( \Rightarrow \) last equality, but only for \( p \) bigger than an unknown bound on root system.
QEAs: Character Formulas

\[
\begin{align*}
\{ \text{quantum group representations at root of unity for }& \mathbb{Q}(q) \} \quad & \overset{KL}{\longleftrightarrow} \quad \{ \text{some representations of affine Lie algebras at negative level } & - p - h \} \\
\quad & \overset{NT}{\longleftrightarrow} \quad \{ \text{some perverse sheaves on } & G_C / B \} 
\end{align*}
\]  \quad (15)

(where the first equivalence arises from a study of representations in the principal block of the quantum group).
THANKS FOR LISTENING!
Acknowledgements: The speaker would like to thank Bhama Srinivasan for her invitation, encouragement, and suggestions for this talk, and Brian Parshall for reviewing a preliminary draft.
For Further Reading

- R. W. Carter and M. Geck, 
  *Representations of Reductive Groups*, 

- J. C. Jantzen, 
  *Representations of Algebraic Groups (2nd Edition)*, 

- R. Steinberg, 
  *Lectures on Chevalley Groups* (typed notes by J. Faulkner; 
  recently put in LaTeX form by C. Drupieski), 
  http://people.virginia.edu/~cmd6a/.

- C. Drupieski and T. Hodge, 
  Irreducible modular representations of finite and algebraic groups. 
Appendix Outline
Proof of Weyl’s Character Formula

(Drawn from [1, Donkin’s paper]) For \( \tau = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) \), recall \( \text{ch}(V(\lambda)) = e^{\lambda + \rho}/\tau \). From this and

\[
\text{ch} L(\lambda)_{\mathbb{C}} = \sum_w a_w \text{ch}(V(w \cdot y)),
\]

observe

\[
\text{ch} L(\lambda)_{\mathbb{C}} = \sum_{w \in W} a^w e^{w \cdot \lambda + \rho}/\tau,
\]

\[
\text{ch} L(\lambda)_{\mathbb{C}} = \sum_{w \in W} a^w e^{w(\lambda + \rho)}/\tau,
\]

so

\[
\tau \text{ch} L(\lambda)_{\mathbb{C}} = \sum_{w \in W} a^w e^{w\lambda + \rho}.
\]
Now, \( \text{ch } L(\lambda)_C \in \mathbb{Z}[\Lambda]^W \), and \( \forall y \in W, \ y\tau = \text{sign}(y)\tau \), since the action of \( y \) on \( \tau \) sends \( (e^{\alpha/2} - e^{-\alpha/2}) \) to its negative \( \ell(y) \) many times in the expression \( \tau \). Thus

\[
y\tau \cdot \text{ch } L(\lambda)_C = y \sum_{w \in W} a_w e^{w(\lambda + \rho)} = \sum_{w \in W} a_w e^{yw(\lambda + \rho)} \quad (16)
\]

and

\[
\text{sign}(y)\tau \text{ch } L(\lambda)_C = \text{sign}(y) \sum_{w \in W} a_w e^{w(\lambda + \rho)}. \quad (17)
\]

Since \( a_1 = 1 \) in (17), the coefficient of \( e^{\lambda + \rho} \) is \( \text{sign}(y) \). OTOH, by shifting indices in (16), the coefficient of \( e^{\lambda + \rho} \) is \( a_y \). Thus \( a_y = \text{sign}(y) = (-1)^{\ell(y)} \), proving

\[
\text{ch } L(\lambda)_C = \sum_{w \in W} (-1)^{\ell(y)} \text{ch}(V(w \cdot \lambda)).
\]
Each simple $G$-module $L(\lambda)$ remains simple on restriction to the finite Chevalley group $G(q)$. Steinberg showed that in fact every irreducible $G(q)$-module can be obtained in this manner. (His result also holds for any finite group $G^{F_{gen}}$ of Lie type, $F_{gen}$ as discussed above.)

**Theorem (Steinberg, 1963)**

Let $L$ be an irreducible module over $k$ for the finite group $G(q)$. Then $L$ is the restriction from $G$ of an irreducible $G$-module.
On the other hand,

- distinct irreducible $G$-modules may no longer be non-isomorphic on restriction to $G(q)$: e.g., for $\lambda \in X_1(T)$, (STPT) $\Rightarrow L(p^r \lambda) \cong L(\lambda)[r]$.

- But $G(q) = G^{Fr}$ is the fixed point subgroup of $G$ under the $r$-th Frobenius morphism, so $G(q)$ doesn’t “see” the twist on $L(\lambda)$ and we have $L(\lambda) \cong L(p^r \lambda)$ as $G(q)$-modules.

To parametrize the simple $G(q)$-modules, we must then restrict our attention to some subset of the dominant weights.

- Steinberg showed that the necessary dominant weights are precisely the $r$-th restricted dominant weights $\lambda \in X_r(\lambda)$. (He also gives a precise description of the weights needed in the general finite group of Lie type case. We stick to the Chevalley groups here and afterwards for simplicity.)

- By the Tensor Product Theorem (next slide), one may even restrict attention to the restricted weights $\lambda \in X_1(T)$.
Theorem

Let $\lambda \in X(T)^+$ and write $\lambda = \sum_{i=0}^{m} p^i \lambda_i$ with $\lambda_i \in X_1(T)$. Then $L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)[^1] \otimes \cdots \otimes L(\lambda_m)[^m]$, where $L(\lambda_j)[^j]$ denotes the $G$-module obtained by composing the structure map for $L(\lambda_j)$ with the $j$-th Frobenius morphism.

In principle then, the structures of the irreducible $G$-modules $L(\lambda)$ are completely determined by those $L(\lambda)$ with restricted weights $\lambda \in X_1(T)$ and by the Frobenius morphism $F : G \rightarrow G$. 

Return to Main Presentation
Note: This term often just refers to the last two relations given below.
Quantum Serre Relations for QEAs

Note: This term often just refers to the last two relations given below.