Modular Representations of Algebraic Groups: or to Characteristic Zero and Back Again, with Applications to Representations of Finite Groups of Lie Type in the Defining Characteristic

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Outline

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1. Characteristic Zero Lie Theory
   - Complex S.s. Lie Algebras and Their Irreducible Modules
   - Character and Dimension Formulae
2. Algebraic Groups in Positive Characteristic
   - A Few Basics
   - Chevalley Groups
   - Frobenius Morphisms
   - Representations of Algebraic Groups
3. Lusztig Conjecture
   - Quantum Enveloping Algebras
   - Representations and Lusztig's Conjecture

Outline

Set-up

- $\mathfrak{g}_{\mathbb{C}}$ complex s.s. Lie algebra, with Cartan subalgebra $\mathfrak{h}$, root system $\Phi \subset \mathfrak{h}_{\mathbb{C}}^{*}$, with Weyl group W and base of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$
- ϖ_i the fundamental dominant (integral) weight corresponding to simple root α_i, weight lattice Λ with partial order $\leq$, root lattice Λ_r, dominant (integral) weights λ^+, $\rho := \frac{1}{2}_{\alpha\in\Phi^+} = \sum_{i=1}^{\ell} \varpi_i$
- Universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$; can be defined as associative algebra (w/1) on generators $e_{\alpha_i}, f_{\alpha_i}, \alpha_i \in \Pi, h_j, j = 1, \ldots, \dim(\Lambda)$, satisfying the Serre relations ▸ Details
- Associated to base Π, triangular decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_{\mathbb{C}}^{-} \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}^{+} \cong \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}^{+}$ (for $\mathfrak{b}_{\mathbb{C}}^{+} := \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}^{+}; \mathfrak{n}_{\mathbb{C}}^{-} := \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}^{-}$) with corresponding triangular decomposition (v. space isos.) $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \cong \mathcal{U}(\mathfrak{n}_{\mathbb{C}}^{-}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{n}_{\mathbb{C}}) \cong \mathcal{U}(\mathfrak{n}_{\mathbb{C}}^{-}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b}_{\mathbb{C}}^{+})$

Irreducible $\mathfrak{g}$-modules

- Given $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, have Verma (or standard) module $V(\lambda)$, a $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ.
- $V(\lambda) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{b}_{\mathbb{C}}^+)} \mathbb{C}_\lambda$, for $\mathbb{C}_\lambda$ 1-dl. $\mathfrak{b}^+$ rep. w/basis v_λ satisfying $\mathfrak{n}^+.v_\lambda = 0$, and $h.v_\lambda = h(\lambda)v_\lambda \forall h \in \mathfrak{h}_{\mathbb{C}}$.
- $V(\lambda) = \mathcal{U}(\mathfrak{n}_{\mathbb{C}}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as v. spaces, so is infinite-dimensional.
- $V(\lambda)$ has weight space decomposition $V(\lambda) = \oplus_{\mu \in \mathfrak{h}_{\mathbb{C}}^*} V(\lambda)_\mu$, $\mu \leq \lambda$. Although there are infinitely many weights, $\dim(V(\lambda)_\mu) < \infty \quad \forall \mu$.
- Every $\mathfrak{g}_{\mathbb{C}}$-module of highest weight λ is a homomorphic image of $V(\lambda)$.
- $V(\lambda)$ has a unique maximal submodule and irreducible head $L(\lambda)_{\mathbb{C}}$; $\dim((L(\lambda)_{\mathbb{C}})_\lambda) = 1$.
- $L(\lambda)_{\mathbb{C}}$ is finite dimensional $\Leftrightarrow \lambda \in \Lambda^+$.
- Consequently, the finite dimensional irreducible $\mathfrak{g}_{\mathbb{C}}$-modules are parameterized (up to isomorphism) by their highest weights, and $\{L(\lambda)_{\mathbb{C}} \mid \lambda \in \Lambda^+\}$ is a representative list of all irreducible finite-dimensional $\mathfrak{g}_{\mathbb{C}}$-modules.

Weight Structures of Verma Modules and Irreducibles, $\lambda \in \Lambda^+$

In terms of weight spaces,

$$V(\lambda) = \begin{array}{|c|}\hline V(\lambda)_\lambda \\ \hline V(\lambda)_\mu \\ \hline \vdots \\ \hline \vdots \\ \hline \vdots \\ \hline V(\lambda)_\nu \\ \hline \vdots \\ \hline \end{array}, \qquad \lambda > \mu, \ldots, \nu, \cdots \Rightarrow \begin{array}{|c|}\hline L(\lambda)_\lambda \\ \hline L(\lambda)_\eta \\ \hline \vdots \\ \hline \vdots \\ \hline L(\lambda)_\zeta \\ \hline \end{array}, \qquad \lambda > \eta, \ldots, \zeta.$$

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Characters, Weyl Modules, and Irreducible Modules

- For any $\mathfrak{g}_{\mathbb{C}}$-module V which is a direct sum $V = \bigoplus_{\mu\in\mathfrak{h}^*}$ of fin. dl. $\mathfrak{h}_{\mathbb{C}}$-weight spaces, one has the character $\mathrm{ch}(V) : \mathfrak{h}^*_{\mathbb{C}} \to \mathbb{Z}$, $\mathrm{ch}(V)(\mu) = \dim(V_\mu)$ (& formal character $\mathrm{ch}(V) = \sum_{\mu\in\mathfrak{h}^*_{\mathbb{C}}}(\dim(V_\mu)e^\mu)$
- For example, $V(\lambda) = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ as v. spaces $\Rightarrow$ $\dim(V(\lambda)_\mu) = \#$ ways to write μ as $\lambda - \sum_{\alpha_i\in\Phi^+} m_i\alpha_i$ ($\lambda-$a non-neg. sum integral sum of pos. roots); consequently, can show $\mathrm{ch}(V(\lambda)) = e^\lambda/\Pi_{\alpha>0}(1-e^{-\alpha}) = e^{\lambda+\rho}/(\Pi_{\alpha>0}(e^{\alpha/2}-e^{-\alpha/2}))$.
- From $\dim(L(\lambda)) < \infty$, $\dim((L(\lambda)_{\mathbb{C}})_\lambda) = 1$, and weight structure, can show the $\mathrm{ch}(L(\lambda)_{\mathbb{C}})$, $\lambda \in \Lambda^+$ form a basis for $\mathbb{Z}[(\Lambda)]^W$, where W acts via $w.e^\mu = e^{w\mu}$.
- If $\dim(V) < \infty$, then $\mathrm{ch}(V) \in \mathbb{Z}[(\Lambda)]^W$, so is determined by $\mathrm{ch}(L(\lambda)_{\mathbb{C}})$, $\lambda \in \Lambda^+$. (Here, moreover, Weyl's Complete Reducibility Thm. $\Rightarrow$ $V =$ a direct sum of some $L(\lambda)_{\mathbb{C}}$s, $\lambda \in \Lambda^+$, so certainly all finite dimensional V are determined by knowledge of $\mathrm{ch}(V)$.)

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Characters, Weyl modules, and Irreducible Modules

Let $\lambda \in \Lambda^+$.

- KEY FACT: $V(\lambda)$ has finite comp. series w/factors $L(\mu)$, and multiplicity $[V(\lambda) : L(\mu)] \neq 0 \Leftrightarrow \mu = w \cdot \lambda \quad \exists w \in W$. Necessarily, $\mu < \lambda$; also recall $[V(\lambda)_{\mathbb{C}} : L(\lambda)_{\mathbb{C}}] = 1$.
- Thus $\mathrm{ch}(V(\lambda)) = \sum_{w \in W} a_w \,\mathrm{ch}(L(w \cdot \lambda)_{\mathbb{C}})$, $a_w \in \mathbb{Z}^{\geq 0}$, $a_1 = 1$.
- Likewise, for $w \cdot \lambda \leq \lambda$, $\mathrm{ch}(V(w \cdot \lambda)) = \sum_{y \in W} a_y \,\mathrm{ch}(L(y \cdot \lambda)_{\mathbb{C}})$, $y \cdot \lambda \leq w \cdot \lambda$.

$$V(\lambda) = \begin{array}{|c|}\hline L(\lambda) \\ \hline L(\mu) \\ \hline \vdots \\ \hline L(w\cdot\lambda) \\ \hline \vdots \\ \hline L(\nu) \\ \hline \end{array}, \qquad \mu, \ldots, w \cdot \lambda, \ldots, \nu < \lambda, w \in W, w \cdot \lambda := w(\lambda + \rho) - \rho.$$

Weyl's Character and Dimension Formulae

- From system of equations for the $\mathrm{ch}(V(w\cdot\lambda))$ in terms of the $\mathrm{ch}(L(y\cdot\lambda)_{\mathbb{C}})$, can order $\{w\cdot\lambda \mid w\cdot\lambda\leq\lambda\}$ to get square upper triangular $\mathbb{Z}$-matrix A with $\mathrm{diag}(A)=(1,1,\ldots,1)$. Then A^{-1} produces an equation

$$\mathrm{ch}(L(\lambda)_{\mathbb{C}})=\sum_{w\in W} b_w\,\mathrm{ch}(V(w\cdot\lambda)) \qquad \exists b_w\in\mathbb{Z}.$$

- Using formula for $\mathrm{ch}(V(\mu))$ and examining Weyl group action on each side of equation above yields Weyl's Character Formula

$$\mathrm{ch}\,L(\lambda)_{\mathbb{C}}=\sum_{w\in W}(-1)^{\ell(w)}\,\mathrm{ch}(V(w\cdot\lambda))$$

▸ Proof details

- The $\mathrm{ch}(V(w\cdot\lambda))$ are known!
- So $\mathrm{ch}(L(\lambda)_{\mathbb{C}}), \lambda\in\Lambda^+$ are all known,
- so $\mathrm{ch}(V)$ for any finite dl. $\mathfrak{g}$-module V is known.
- From Weyl's Character Formula, one also obtains $\dim(L(\lambda)_{\mathbb{C}})\ \forall\lambda\in\Lambda^+$.

Outline

A Few Basics about Affine Algebraic Groups

Set $k = \overline{\mathbb{F}_p}$.

- An affine (i.e., linear) algebraic group G over k can be viewed as an affine algebraic variety $G \subset k^n$ $(\exists n)$ with a compatible group structure. Have Borel B, maximal torus T, characters $X(T)$, cocharacters $Y(T)$, dominant weights $X(T)^+$...
- Coordinate algebra $k[G]$ of such a variety is a finitely generated reduced commutative k-algebra. It is also a Hopf algebra.
- Can also consider G functorially as a representable functor from category of commutative k-algebras to category of groups, with $G(A) = \text{Hom}_{k-alg}(k[G], A)$, so that $G(k)$ identifies with G originally regarded as an affine algebraic variety.
- Can expand functorial perspective to use representing algebras $R \in k - alg$ in place of $k[G]$ which are fin. gen., but not necessarily reduced ("algebraic affine k-group schemes") or just commutative, but not even fin. gen. ("affine k-group scheme"). If R is not just fin. generated, but f. dl., call $\text{Hom}_{k-alg}(R, -)$ finite; includes "infinitesimal group schemes".

- A linear algebraic group G is a closed subgroup of $GL_n(k)$ for some n. There is a natural notion of a (rational) G-module V, e.g., alg. group hom $G \to GL(V)$ (by assumption, $\dim(V) < \infty$). There is a compatible notion of G-modules for group schemes via group scheme maps $G \to GL(V)$, or comodules for the Hopf algebra $k[G]$.
- Complete reducibility for (rational, f. dl.) G-modules does not hold. (Consider $SL_2(k)$ acting on symmetric powers $S^i(V)$, $i = p^r$, for natural module $V = k^2$ with standard basis u, v as an example.)

Outline

- $\mathfrak{g}_{\mathbb{C}}$ complex s.s. Lie algebra with Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, weight lattice Λ, root lattice Λ_r.
- For faithful finite-dl. $\mathfrak{g}_{\mathbb{C}}$ module V, Λ_V sublattice of Λ_r generated by all weights of $\mathfrak{h}_{\mathbb{C}}$ on V, have $\Lambda_r \subset \Lambda_V \subset \Lambda$.
- $\{X_\alpha \in \mathfrak{g}_{\mathbb{C}} : \alpha \in \Phi\} \cup \{H_i \in \mathfrak{h}_{\mathbb{C}} : \alpha_i \in \Pi\}$ Chevalley basis for $\mathfrak{g}_{\mathbb{C}}$.
- $\mathcal{U}_{\mathbb{Z}}$ Kostant $\mathbb{Z}$-form of enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, subalgebra generated by all $X_\alpha^{(n)} := \frac{X_\alpha^n}{n!}$, $\alpha \in \Phi, n \in \mathbb{N}$.
- $\exists$ lattice $V_{\mathbb{Z}}$ in V invariant under $\mathcal{U}_{\mathbb{Z}}$; for $k = \overline{\mathbb{F}_p}$, set $V_k := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$.
- For $t \in k, \alpha \in \Phi$, $\exp(tX_\alpha) : V_k \to V_k$ defined by $\exp(tX_\alpha)(v \otimes a) = \sum_{n=0}^{\infty} .v \otimes t^n a$ is well-defined automorphism.
- Set G to be subgroup of $\mathrm{Aut}\,(V_k)$ generated by all $\exp(tX_\alpha)$, $t \in k, \alpha \in \Phi$.
- By def., G is a Chevalley group; is s.s. alg. group defined over $\mathbb{F}_p$ w/$\mathfrak{g} := \mathrm{Lie}\,(G) = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ for $\mathfrak{g}_{\mathbb{Z}}$ the lattice in $\mathfrak{g}_{\mathbb{C}}$ preserving the $\mathbb{Z}$-form $V_{\mathbb{Z}}$.
- Chevalley group G has maximal torus T with $X(T) = \Lambda_V$, root lattice Λ_r and weight lattice Λ w.r.t. T.
- Chevalley group G is universal if $\Lambda_V = \Lambda$ iff G is simply connected.

Chevalley Groups, the Kostant $\mathbb{Z}$-form, and Hyperalgebras

$$
\begin{array}{ccccccc}
\mathrm{Dist}(G_{\mathbb{C}}) & & & & \mathcal{U}(\mathfrak{g}_k) & \twoheadrightarrow & \mathcal{U}^{[p]}(\mathfrak{g}_k) \\
\| & & & & \downarrow & & \\
\mathcal{U}_{\mathbb{Z}} \subset \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) & \longleftarrow & \mathcal{U}_{\mathbb{Z}} & \dashrightarrow & \mathrm{Dist}(G) = \mathcal{U}_{\mathbb{Z}} \otimes_k k = \mathcal{U}_k & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathfrak{g}_{\mathbb{C}} & \longleftarrow & \mathfrak{g}_{\mathbb{Z}} & \dashrightarrow & \mathfrak{g}_k = \mathfrak{g}_{\mathbb{Z}} \otimes_k k = \mathrm{Lie}(G) & & \\
\vdots & & & & \vdots & & \\
G_{\mathbb{C}} = G(V_{\mathbb{C}}, \mathbb{C}) \subset GL(V_{\mathbb{C}}) & & & & G = G(V_k, k) \subset GL(V_k) & &
\end{array}
$$

$$
V_{\mathbb{C}} \longleftarrow V_{\mathbb{Z}} \dashrightarrow V_k = V_{\mathbb{Z}} \otimes_k k
$$

Outline

Frobenius Morphisms and Finite Groups of Lie Type

- $k = \overline{\mathbb{F}_p}$
- G connected affine algebraic group defined over $\mathbb{F}_p \subset k$ (so there is a Hopf algebra A_0 such that $k[G] \cong k \otimes_{\mathbb{F}_p} A_0$)
- $F : G \to G$ Frobenius morphism (induced by comorphism $F^* : k[G] \to k[G], (\alpha \otimes f) \mapsto \alpha \otimes f^p$)
- r^{th} Frobenius morphism F^r ($= r^{th}$ power of F), $r \geq 1$
- For $q = p^r$, $G(\mathbb{F}_q) := G^{F^r} = \{g \in G \,|\, F^r(g) = g\}$, finite group of $\mathbb{F}_q$ rational points of G
- More generally, can consider ‘generalized Frobenius morphisms’ $F_{gen} : G \to G$, characterized by $F_{gen}^m = F^r$ for some $m, r \geq 1$
- Every finite group of Lie type arises as $G^{F_{gen}}$ for some generalized Frobenius morphism.

Frobenius Morphisms and Frobenius Kernels

- $k = \overline{\mathbb{F}_p}$, G/k affine algebraic group defined over $\mathbb{F}_p$
- $F : G \to G$ Frobenius morphism (induced by comorphism $F^* : k[G] \to k[G]$, $(\alpha \otimes f) \mapsto \alpha \otimes f^p$)
- r^{th} Frobenius morphism F^r ($= r^{th}$ power of F), $r \geq 1$
- $G_r = \ker(F^r)$, r^{th} Frobenius kernel, normal subgroup of G
- G_r is an infinitesimal group scheme, a 'nontrivial trivial group': $G_r(K) = \text{Hom}_{k-alg}(k[G_r], K) = \{e\}$, the trivial group, for any field extension $K \supset k$.

Chevalley Groups, the Kostant $\mathbb{Z}$-form, Hyperalgebras and Frobenius Kernels

▸ Go to QEA Case

$$
\begin{array}{ccccccc}
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\| & & & & \downarrow & & \| \\
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\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathfrak{g}_{\mathbb{C}} & \twoheadleftarrow & \mathfrak{g}_{\mathbb{Z}} & \dashrightarrow & \mathfrak{g}_k = \mathfrak{g}_{\mathbb{Z}} \otimes_k k = \mathrm{Lie}(G) & = & \mathrm{Lie}(G_1) \\
\vdots & & & & \vdots & & \\
G_{\mathbb{C}} = G(V_{\mathbb{C}}, \mathbb{C}) \subset GL(V_{\mathbb{C}}) & & & & G = G(V_k, k) \subset GL(V_k) & &
\end{array}
$$

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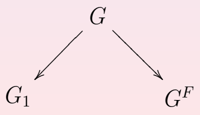
- Universal Chevalley group G/k, defined over $\mathbb{F}_p \subset k = \overline{\mathbb{F}_p}$
- $F : G \to G$ Frobenius morphism (and $F_{gen} : G \to G$ generalized Frob. morphism)
- By Curtis and Steinberg, the irreducible representations of a finite group $G^{F_{gen}}$ of Lie type in *defining characteristic* are 'determined' by the representation theory of the overarching algebraic group G.
  ▸ Details
- By Steinberg's Tensor Product Theorem, the irreducible representations for G in positive characteristic p are 'determined' by the representation theory of the Frobenius kernel $G_1 = \ker(F)$.
  ▸ Details

GOAL: Determine characters and dimensions of $L(\lambda)$, $\lambda \in X_1(T)^+$.



Comparisons: Reps. for $\mathfrak{g}_{\mathbb{C}}$ and G

$$\left\{\begin{array}{l}\text{Complex s.s. Lie alge-}\\ \text{bra } \mathfrak{g}_{\mathbb{C}}\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{S.s. s. conn. algebraic group } G/k\\ \text{def. and split over } \mathbb{F}_p \subset k \text{ (univer-}\\ \text{sal Chevalley group assoc. to } \mathfrak{g})\end{array}\right\} \tag{1}$$

Standard Modules

$$\{\text{Verma module } V(\lambda)\} \leftrightsquigarrow \left\{\begin{array}{l}\text{Weyl module } \Delta(\lambda) := \\ (L(\lambda)_{\mathbb{C}})_{\mathbb{Z}} \otimes_{\mathbb{Z}} k\end{array}\right\} \tag{2}$$

Costandard Modules

$$\{\text{Dual Verma module}\} \leftrightsquigarrow \left\{\begin{array}{l}\text{Induced module}\\ \nabla(\lambda) = H^0(\lambda)\end{array}\right\} \tag{3}$$

Irreducible Finite-Dim'l Modules (λ dominant)

$$\{L(\lambda)_{\mathbb{C}}\} \leftrightsquigarrow \left\{\begin{array}{l}L(\lambda) := \\ head(\Delta(\lambda))\end{array}\right\} \tag{4}$$

- PROBLEM: $\Delta(\lambda)$ is not irreducible.
- FIX: From weight structures, and Weyl's Character Formula, still get picture below, so as before, get $\mathrm{ch}(L(\lambda)) = \sum_\mu b_{\mu,\lambda} \,\mathrm{ch}\, \Delta(\mu)$, $b_{\lambda,\lambda} = 1$. Then, solve for $b_{\mu,\lambda}$!
- PROBLEM: Proof of Weyl's Char. Formula (really, any proof) makes use of $[V(\lambda) : L(\mu)_{\mathbb{C}}] \neq 0 \Rightarrow \mu = w \cdot \lambda$, not just $\mu < \lambda$.

  Proof of Weyl's Char. Formula
- Linkage Principle: $[\Delta(\lambda), L(\mu)] \neq 0 \Rightarrow \mu \in W \cdot \lambda + p\mathbb{Z}\Phi$.

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- Linkage Principle: $[\Delta(\lambda), L(\mu)] \neq 0 \Rightarrow \mu \in W \cdot \lambda + p\mathbb{Z}\Phi$.

Character Theories: $\mathfrak{g}_{\mathbb{C}}$ and G

$$\left\{\begin{array}{l}\text{Complex s.s. Lie algebra } \mathfrak{g}_{\mathbb{C}}\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{universal Chevalley group } G/k, \\ \text{char}(k) = p > 0\end{array}\right\} \tag{5}$$

Characters

$$\left\{\begin{array}{l}\text{ch } L(\lambda)_{\mathbb{C}} = \sum_{w\in W} b_w \text{ ch } V(w\cdot\lambda) \\ \text{for some } b_w \in \mathbb{Z}\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{ch } L(\lambda) = \sum_{\substack{w\in W_p \\ w\cdot\lambda\in X^+}} b_w \text{ ch } \Delta(w\cdot\lambda) \\ \text{for some } b_w \in \mathbb{Z}, \\ W_p = p\mathbb{Z}\Phi \rtimes W\end{array}\right\} \tag{6}$$

$$\left\{\begin{array}{l}\text{ch}(L(\lambda)_{\mathbb{C}}) \text{ known since ch } V(w\cdot\lambda) \text{ known,} \\ \text{and } b_w = (-1)^{\ell(\lambda)} \\ \text{(Weyl's Char. Form.)}\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{ch}(\Delta(\lambda) = L(\lambda)_{\mathbb{Z}} \otimes k) \text{ known} \\ \text{(via Weyl's Char. Form.),} \\ \text{but } b_w \text{ `known' only up to} \\ \text{Lusztig's conjecture}\end{array}\right\} \tag{7}$$

LC for Algebraic Groups in Positive Characteristic

Assume G is a universal Chevalley group constructed from complex s.s. Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

- Lusztig's conjecture, below, asserts that the coefficients b_w are in effect given by the values at 1 of certain polynomials $P_{y,w}$, called Kazhdan-Lusztig polynomials, associated with the Coxeter group W_p.
- Though Lusztig's formula is known to be correct for $p \gg h$, where $h = 1 + \langle \rho, \alpha_0^\vee \rangle$ is the Coxeter number of Φ (α_0 is the longest short root of Φ), a lower bound for p is not known.
- Additional terminology: we say that a dominant weight μ lies in the Jantzen region if $\langle \mu + \rho, \alpha_0^\vee \rangle \leq p(p - h + 2)$.

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LC for Algebraic Groups in Positive Characteristic

Conjecture (Lusztig, 1979)

Let λ be a weight in the Jantzen region (which includes all restricted weights if $p \geq 2h - 2$, h the Coxeter number of Φ). Then if $p \geq h$, $\dim L(\lambda)_\nu$ is given as follows: Choose w in the affine Weyl group $W_p = p\mathbb{Z}\Phi \rtimes W$ such that $\lambda = w \cdot \lambda_0$, for some λ_0 (unique) with $-p \leq (\lambda_0 + \rho)(H_\alpha) \leq 0$ for all $\alpha \in \Phi^+$. (We say that λ_0 is in the antidominant lowest alcove.) Let w_0 denote the longest element of W. Then

$$\dim L(\lambda)_\gamma = \sum (-1)^{l(w)-l(y)} P_{y,w}(1) \dim \Delta(w_0 y \cdot \lambda_0)_\gamma \tag{8}$$

where the sum is taken over all $y \in W$ such that $w_0 y \cdot \lambda_0$ is dominant and $w_0 y \cdot \lambda_0 \leq w_0 w \cdot \lambda_0 = \lambda$, $\Delta(w_0 y \cdot \lambda_0)$ is the Weyl module of highest weight $w_0 y \cdot \lambda_0$, and $P_{y,w}$ is a Kazhdan-Lusztig polynomial associated with the Coxeter group W_p.

Outline

Set-up

- For indeterminate v, $\mathbb{U}_v = \mathbb{U}_v(\mathfrak{R})$ quantum (or quantized) enveloping algebra (QEA) (a.k.a. 'quantum group') associated to root datum $\mathfrak{R} = (\Pi, X, \Pi^\vee, X^\vee)$ is a $\mathbb{Q}(v)$-algebra (w/1) defined by generators $E_i, F_i, i \leftrightarrow \alpha_i \in \Pi, K_h, h \in X^\vee$, subject to the quantum Serre relations involving v Details
- For $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, there is Lusztig integral form $\mathbb{U}_{\mathcal{A}}$ of $\mathbb{U}_v$, an $\mathcal{A}$-subalgebra generated by appropriately defined 'divided powers' $E_i^{(n)}, F_i^{(n)}, n \in \mathbb{Z}^{\geq 0}$ along with certain expressions involving the K_hs (see e.g., [2, H.5]).
- For any commutative ring K and any invertible element $q \in K$, there are specializations $\mathbb{U}_{q,K} := \mathbb{U}_{\mathcal{A}} \otimes_{\mathcal{A}} K$ arising from the $\mathcal{A}$-module structure on K given by the (unique) ring homomorphism $\epsilon_q : \mathcal{A} \to K$, $v \mapsto q$.
- For each algebra above, there is an associated triangular decomposition, with $+$-part associated to E_i-type generators, $-$-part associated to F_i-type generators, and 0-part associated to appropriate K_h expressions.

From here on, assume $\mathfrak{R}$ is root datum of a simply connected, semisimple algebraic group G/k, defined over $\mathbb{F}_p \subset k = \overline{\mathbb{F}_p}$. Also, let $K := \mathbb{Q}(q)$, p^{th} cyclotomic field (so q is a prim. p^{th}-root of unity). Then

- $\mathbb{U}_{q,K}$ is generated over K by all $E_i, E_i^{(p)}, F_i, F_i^{(p)}, K_h^{\pm 1}$
- 'Small quantum group' $\mathbf{u}_{q,K} \subset \mathbb{U}_{q,K}$ is subalgebra generated by all $E_i, F_i, K_h^{\pm 1}$; has finite dimension $\dim(\mathbf{u}_{q,K}) = 2^{|\Pi|} p^{\dim(G)}$.

▸ Compare w/Alg. Gps.

$$\begin{array}{ccccccc}
\mathbb{U}_v := \mathbb{U}_v(\mathfrak{R}) & \longleftarrow\!\!\!\ll & \mathbb{U}_{\mathcal{A}} & - - - \rightsquigarrow & \mathbb{U}_{q,K} := \mathbb{U}_{\mathcal{A}} \otimes_{\mathcal{A}} K(\text{ via } \epsilon_q) & \longleftarrow\!\!\!\ll & \mathbf{u}_{q,K} \\
\| & & \| & & \| & & \| \\
\mathbb{U}_v^- \otimes_{\mathbb{Q}(v)} \mathbb{U}_v^0 \otimes_{\mathbb{Q}(v)} \mathbb{U}_v^+ & \longleftarrow\!\!\!\ll & \mathbb{U}_{\mathcal{A}}^- \otimes_{\mathcal{A}} \mathbb{U}_{\mathcal{A}}^0 \otimes_{\mathcal{A}} \mathbb{U}_{\mathcal{A}}^+ & - - - \rightsquigarrow & \mathbb{U}_{q,K}^- \otimes_K \mathbb{U}_{q,K}^0 \otimes_K \mathbb{U}_{q,K}^+ & \longleftarrow\!\!\!\ll & \mathbf{u}_{q,K}^- \otimes_K \mathbf{u}_{q,K}^0 \otimes_K \mathbf{u}_{q,K}^+
\end{array}$$

Outline

Comparisons: QEAs[1] and Algebraic Groups

$$\left\{\begin{array}{l}\text{QEA } U_{q,K}(\mathfrak{R}),\ q\ \ell^{th}\text{-}\\ \text{root of unity in field } K,\\ \text{char }(K)=0\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{Universal Chevalley group}\\ G/k \text{ of root datum } \mathfrak{R},\\ \text{char }(k)=p>0\end{array}\right\} \tag{9}$$

Standard Modules

$$\left\{\begin{array}{l}\text{quantum Verma mod-}\\ \text{ule } V_q(\lambda)\end{array}\right\} \leftrightsquigarrow \{\text{Weyl module } \Delta(\lambda)\} \tag{10}$$

Costandard Modules

$$\left\{\begin{array}{l}\text{quantum induced}\\ \text{module } H^0_q(\lambda)\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{Induced module}\\ \nabla(\lambda)=H^0(\lambda)\end{array}\right\} \tag{11}$$

Irreducible Finite-Dim'l Modules

$$\{L_q(\lambda)\} \leftrightsquigarrow \{L(\lambda)\} \tag{12}$$

[1] WLOG, modules are integrable type 1

Character Theories: QEAs and Algebraic Groups

$$\left\{\begin{array}{l}\text{QEA } U_{q,K}(\mathfrak{R}),\ q\ \ell^{th}\text{-}\\ \text{root of unity in field}\\ K,\ \text{char}\,(K)=0\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{l}\text{universal Chevalley}\\ \text{group } G/k,\ \text{root da-}\\ \text{tum } \mathfrak{R},\ \text{char}\ (k) =\\ p>0\end{array}\right\} \tag{13}$$

<u>Characters</u>

$$\left\{\begin{array}{l}\text{ch}\, L_q(\lambda)\ \text{known}\\ \text{(Lusztig's Conjecture}\\ \text{(8) with } L_q(\lambda),\ \text{resp.,}\\ \nabla_q(\lambda)\ \text{instead of } L(\lambda),\\ \text{resp., } \nabla(\lambda))\ \text{is a theo-}\\ \text{rem}^2)\end{array}\right\} \overset{[AJS]}{\leftrightsquigarrow} \left\{\begin{array}{l}\text{ch}\, L(\lambda)\ \text{known}\\ \text{(via Lusztig's Conj.}\\ \text{for QEA's),}\\ \text{but w/out precise}\\ \text{lower bound on } p\end{array}\right\} \tag{14}$$

[2] With a few limitations.

Some Steps in Proof of LC for Algebraic Groups

Reduction from algebraic to quantum group case: Assume $p \geq 2h - 2$.

$$\text{ch}\, L(\mu) = \text{ch}\, L_q(\mu)$$

$$\Leftrightarrow \text{ch}\, \tilde{L_1}(\mu) = \text{ch}\, \tilde{L_q}(\mu)$$

for LHS = simple $G_1 T$-module, RHS = simple $\mathbf{u}_{q,k}\mathbb{U}^0_{q,K}$-module "of type 1" (i.e., K_h^p acts as 1)

$$\Leftrightarrow [\tilde{Z}_1(\mu) : \tilde{L}_1(\nu)] = [\tilde{Z}_q(\mu) : \tilde{L}_q(\nu)] \forall \mu, \nu \in X(T),$$

for $\tilde{Z}_1(\mu)$ induction from $B_1 T$ to $G_1 T$ and $\tilde{Z}_q(\mu)$ induction from $\mathbf{u}^-_{q,k}\mathbb{U}^0_{q,K}$ to $\mathbf{u}_{q,k}\mathbb{U}^0_{q,K}$

$$\Leftrightarrow \text{ch}\, \tilde{Q}_1(\lambda) = \text{ch}\, \tilde{Q}_q(\lambda) \forall \lambda \in X(T),$$

for LHS module = injective hull of $\tilde{L}_1(\lambda)$ in category of $G_1 T$-modules, and RHS module = injective hull in category of $\mathbf{u}_{q,k}\mathbb{U}^0_{q,K}$-modules. Last equality holds iff holds for all $\lambda \in X(T)^+$ with $< \lambda + \rho, \alpha^\vee >< p$ for all $\alpha \in \Pi$.

[AJS] $\Rightarrow$ last equality, but only for p bigger than an unknown bound on root system.

QEAs: Character Formulas

$$\left\{\begin{array}{l}\text{quantum group}\\ \text{representations at}\\ \text{root of unity for}\\ \mathbb{Q}(q)\end{array}\right\} \overset{KL}{\longleftrightarrow} \left\{\begin{array}{l}\text{some representa-}\\ \text{tions of affine Lie}\\ \text{algebras at nega-}\\ \text{tive level } -p-h\end{array}\right\} \overset{NT}{\longleftrightarrow} \left\{\begin{array}{l}\text{some perverse}\\ \text{sheaves on } G_{\mathbb{C}}/B\\ \text{(KL)}\end{array}\right\}, \tag{15}$$

(where the first equivalence arises from a study of representations in the principal block of the quantum group).

The End

THANKS FOR LISTENING!

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For Further Reading

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Representations of Algebraic Groups (2nd Edition),
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R. Steinberg,
Lectures on Chevalley Groups (typed notes by J. Faulkner; recently put in LaTEX form by C. Drupieski),
http://people.virginia.edu/~cmd6a/.

C. Drupieski and T. Hodge,
Irreducible modular representations of finite and algebraic groups.
http://www.aimath.org/pastworkshops/finiteliegps.html (2007).

Appendix Outline

Proof of Weyl's Character Formula

(Drawn from [1, Donkin's paper]) For $\tau = \Pi_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})$, recall $\mathrm{ch}(V(\lambda)) = e^{\lambda+\rho}/\tau$. From this and

$$\mathrm{ch}\, L(\lambda)_{\mathbb{C}} = \sum_{w} a_w \,\mathrm{ch}(V(w \cdot y)),$$

observe

$$\mathrm{ch}\, L(\lambda)_{\mathbb{C}} = \sum_{w \in W} a^w e^{w \cdot \lambda + \rho}/\tau$$

$$\mathrm{ch}\, L(\lambda)_{\mathbb{C}} = \sum_{w \in W} a^w e^{w(\lambda+\rho)}/\tau,$$

so

$$\tau\,\mathrm{ch}\, L(\lambda)_{\mathbb{C}} = \sum_{w \in W} a^w e^{w\lambda+\rho}.$$

Return to Main Presentation (Lie algebras) Return to Main Presentation (Algebraic groups)

Now, $\mathrm{ch}\, L(\lambda)_{\mathbb{C}} \in \mathbb{Z}[\Lambda]^W$, and $\forall y \in W$, $y\tau = sign(y)\tau$, since the action of y on τ sends $(e^{\alpha/2} - e^{-\alpha/2})$ to its negative $\ell(y)$ many times in the expression τ. Thus

$$y\tau.\,\mathrm{ch}\, L(\lambda)_{\mathbb{C}} = y \sum_{w \in W} a_w e^{w(\lambda+\rho)} = \sum_{w \in W} a_w e^{yw(\lambda+\rho)} \tag{16}$$

and

$$sign(y)\tau\, \mathrm{ch}\, L(\lambda)_{\mathbb{C}} = sign(y) \sum_{w \in W} a_w e^{w(\lambda+\rho)}. \tag{17}$$

Since $a_1 = 1$ in (17), the coefficient of $e^{\lambda+\rho}$ is $sign(y)$. OTOH, by shifting indices in (16), the coefficient of $e^{\lambda+\rho}$ is a_y. Thus $a_y = sign(y) = (-1)^{\ell(y)}$, proving

$$\mathrm{ch}\, L(\lambda)_{\mathbb{C}} = \sum_{w \in W} (-1)^{\ell(y)}\, \mathrm{ch}(V(w \cdot \lambda)).$$

Return to Main Presentation (Lie algebras) Return to Main Presentation (Algebraic groups)

Restriction of Modules to G^F

Each simple G-module $L(\lambda)$ remains simple on restriction to the finite Chevalley group $G(q)$.Steinberg showed that in fact every irreducible $G(q)$-module can be obtained in this manner. (His result also holds for any finite group G^{F_gen} of Lie type, F_{gen} as discussed above.)

Theorem (Steinberg, 1963)

Let L be an irreducible module over k for the finite group $G(q)$. Then L is the restriction from G of an irreducible G-module.

On the other hand,

- distinct irreducible G-modules may no longer be non-isomorphic on restriction to $G(q)$: e.g., for $\lambda \in X_1(T)$, (STPT) $\Rightarrow L(p^r\lambda) \cong L(\lambda)^{[r]}$.
- But $G(q) = G^{F^r}$ is the fixed point subgroup of G under the r-th Frobenius morphism, so $G(q)$ doesn't "see" the twist on $L(\lambda)$ and we have $L(\lambda) \cong L(p^r\lambda)$ as $G(q)$-modules.
- To parametrize the simple $G(q)$-modules, we must then restrict our attention to some subset of the dominant weights.
- Steinberg showed that the necessary dominant weights are precisely the r-th restricted dominant weights $\lambda \in X_r(\lambda)$. (He also gives a precise description of the weights needed in the general finite group of Lie type case. We stick to the Chevalley groups here and afterwards for simplicity.)
- By the Tensor Product Theorem (next slide), one may even restrict attention to the restricted weights $\lambda \in X_1(T)$.

Return to Main Presentation
Go to Steinberg's Tensor Product Theorem

Steinberg's Tensor Product Theorem

Theorem

Let $\lambda \in X(T)^+$ and write $\lambda = \sum_{i=0}^{m} p^i \lambda_i$ with $\lambda_i \in X_1(T)$. Then $L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_m)^{[m]}$, where $L(\lambda_j)^{[j]}$ denotes the G-module obtained by composing the structure map for $L(\lambda_j)$ with the j-th Frobenius morphism.

In principle then, the structures of the irreducible G-modules $L(\lambda)$ are completely determined by those $L(\lambda)$ with restricted weights $\lambda \in X_1(T)$ and by the Frobenius morphism $F : G \to G$. Return to Main Presentation

Serre Relations for Universal Enveloping Algebras[3]

Return to Main Presentation

[3]Note: This term often just refers to the last two relations given below.

Quantum Serre Relations for QEAs[4]

Return to Main Presentation

[4] Note: This term often just refers to the last two relations given below.