References

- Articles of T. Hawkins.
- C. W. Curtis: Pioneers of Representation Theory: History of Math., AMS., 1999.
- L.: Representations of Finite Groups: A Hundred Years, I, II., Notices AMS, 1997.

Ferdinand Georg Frobenius (1849–1917)

Early History of Rep. Theory
(Stories and Facts)

Cartan ↓ Molien → Wedderburn Noether

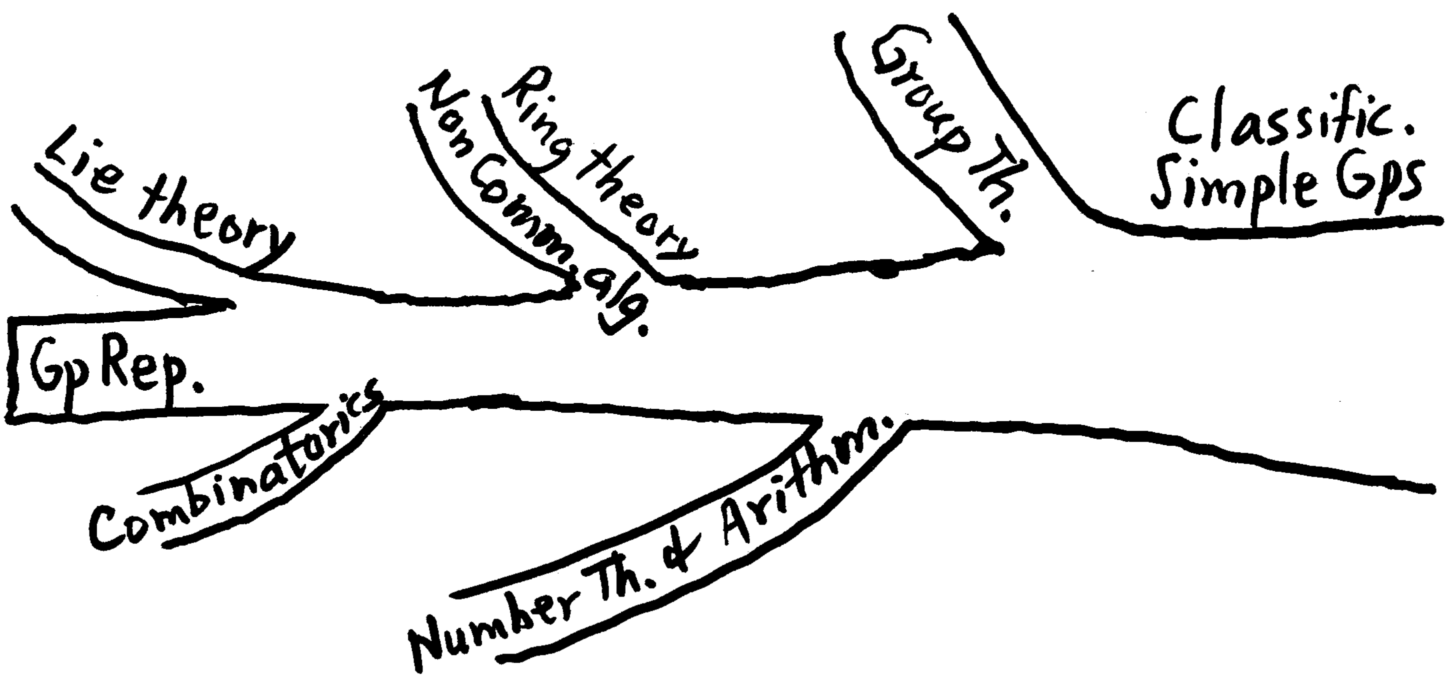
(Dedekind) → Frobenius → Schur → Brauer

Burnside Young

Brauer → { Steinberg ⋮ Feit }

(MacLane) → Thompson

(Zariski) → Gorenstein



Rep. of Finite Groups (On One Page)

Representation: $D : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ (homom.)

Equivalence: $D'(g) = U \cdot D(g) \cdot U^{-1} \quad (\forall g)$

Irred. (Indecom.) D: like an "Atom" ....

Character: $\chi(g) = \mathrm{trace}\,(D(g))$ (determines D !)

Class Function: $\chi(h^{-1}gh) = \chi(g)$

Character Table: $s \times s$ matrix, $s = \#$(conj. classes)

Frob. (1896)

$\mathbf{A_5}$	(1)	(12)(34)	(123)	(12345)	(13524)
60	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	τ	τ'
χ_3	3	-1	0	τ'	τ
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

(Different groups may have the same character table)

"Magic Equation": $|G| = \sum \chi_i(1)^2$

"Frobenius Divisibility": Each $\chi_i(1)$ divides $|G|$

"1st & 2nd $\perp$-Relations": Row & Col. relations

- **Exercise for Fun-loving Folks !**

Call a group *funny* if its irred. rep's have exactly degrees $1, 2, 3, \ldots, n$. Show that all funny groups are trivial.

Characters on Abelian Groups

Legendre (1752–1833) Symbol: $a \mapsto \left(\frac{a}{p}\right) = \pm 1$

Gauss (1777–1855): Binary quad. forms, Gauss $\sum$

Dirichlet (1805–1859) L-series: $L(s, \chi) = \sum \frac{\chi(\overline{n})}{n^s}$

Dedekind (1831–1916) defined (c. 1879) the general notion of characters on abelian groups, $\chi : G \to \dot{\mathbb{C}}$

Factoring the Group Determinant

(Dedekind-Frobenius Briefwechsel: April, 1896)

Take variables $\{ x_g : g \in G\}$, and build the matrix $(x_{gh^{-1}})$. Then form the "group determinant":

$$\Phi = \det \left(x_{gh^{-1}}\right).$$

How to factor Φ *into irreducible factors over* $\mathbb{C}$?

Dedekind: If G is abelian:

$$\Phi = \prod_{\chi \in G^*} \Big(\sum_{k \in G} \chi(k)\, x_k \Big).$$

Here, $\chi : G \longrightarrow \dot{\mathbb{C}}$ runs over the characters of G.

Now, what if G is *not* abelian ??

Example. $G = S_3 = \{x_1, x_2, x_3, \dots\}$. Here

$$\Phi = \begin{vmatrix} x_1 & x_3 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_3 & x_5 & x_6 & x_4 \\ x_3 & x_2 & x_1 & x_6 & x_4 & x_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = \Phi_1 \Phi_2 (\Phi_3)^2;$$

$$\begin{aligned} \Phi_1 &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6 , \\ \Phi_2 &= x_1 + x_2 + x_3 - x_4 - x_5 - x_6 , \\ \Phi_3 &= x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 \\ &\qquad + x_4x_5 + x_4x_6 + x_5x_6 - x_1x_2 - x_1x_3 - x_2x_3 . \end{aligned}$$

Frobenius (1849–1917):

(1896) Factored Φ into irreducibles over $\mathbb{C}$:

$$\Phi = \prod_{i=1}^{r} \Phi_i^{f_i}, \qquad e_i := \deg \Phi_i ; \quad e_i = f_i .$$

Each Φ_i "arises" from a class function $\chi_i : G \to \mathbb{C}$ ($\chi_i(1) = e_i$), which Frobenius called a (higher) **character** of G. Comparison of degrees $\Longrightarrow$ the Magic Equation $|G| = \sum e_i^2$.

(1897) Gave def. of "representation" as we know it today; formula $\chi_D(g) = \sum_i D(g)_{ii}$ appeared for the first time.

Frobenius' Major Contributions

Basic Character Theory: Orthogonality Relations

Frobenius Integrality Thm. $\forall$ irred. χ, $\forall\, g \in G$:
$|\mathrm{Cl}\,(g)| \cdot \chi(g)/\chi(1)$ is an algebraic integer

Induced Representations: Frobenius Reciprocity

Characters of S_n ($\longrightarrow$ Young, Schur, Combinatorics)

Characters of $\mathbf{PSL}_2(p)$ ($\longrightarrow$ Groups of Lie Type)

Frobenius Groups ($\longrightarrow$ Exceptional Characters)

Frobenius' Most Profound Theorem (1901) Let G act transitively on $\{1, ..., n\}$, such that $\pi(g) \leq 1$ for every $g \neq 1$. If $\pi(g) = \pi(h) = 0$, then $\pi(gh) = 0$, unless $gh = 1$.

Proof. Induced representations ! (NO group-theoretic proof ever found — except in some special cases.)

The set $\{g \in G : \pi(g) = 0\}$ together with 1 form a (normal) subgroup $K \lhd G$, called the *Frobenius kernel.*

"Frob. Conj." K is a *nilpotent* group (if $K \neq G$).

Affirmed in 1959: Thompson's Chicago Thesis.

- *If a group K has a fixed-point-free automorphism of prime order p, then K is nilpotent.*

Enters Burnside (1852–1927)

Used Lie algebra structure on $\mathbb{C}G$ and regular repres.; recast def. of irred. representations. etc.

First Group Theory book in English: 1897; 1911.

Preface, 2nd edition (1911):

"..... the reason given in the original preface for omitting any account of it no longer holds good. In fact it is now more true to say that, for further advances in the abstract theory, one must look largely to the representation of a group as a group of linear substitutions."

Burnside's Most Profound Theorem (1904)

- *Any finite group with a conjugacy class of size* p^k $(p = \textit{prime},\ k > 0)$ *is non-simple.*
- *Any group of order* $p^a q^b$ *is solvable.*

The proof of this, using **Frobenius' Integrality** Theorem, represents the beginning of the influx of number theory and arithmetic into repres. theory.

Goldschmidt (1970): p, q odd.

Bender (1972): including $p = 2$.

Matsuyama (1973): specifically for $p = 2$.

"Burnside Conjecture"

"The contrast that these results shew between groups of odd and even order suggests inevitably that non-abelian simple groups of odd order do not exist. ...... The results ... appear to me to indicate that an answer to the interesting question as to the existence or non-existence of simple groups of odd composite order may be arrived at by a further study of the theory of group characteristics."

Feit-Thompson Odd-Order Theorem (1963)

(1) *Odd Groups Are Solvable; or equivalently,*

(2) (*Nonabelian*) *Simple Groups Are Even.*

- Frobenius groups figured prominently.
- Large parts were devoted to character computations.
- 255 pages long!

Wikipedia: "Perhaps the most revolutionary and important new idea was that of the very long paper: before their paper, few arguments in group theory were more than a few pages long ... Once group theorists realized that such long arguments could work, a series of papers that were several hundred pages long started to appear. Some of these even dwarfed the Feit-Thompson paper: Aschbacher and Smith's paper on quasi-thin groups were over 1000 pages long ..."

Influx From Ring Theory

Cartan (1894): classified simple Lie algebras over $\mathbb{C}$. Showed simple (assoc.) algebra $\cong \mathbb{M}_n(\mathbb{C})$.

Molien (1893): simple and semisimple algebras over $\mathbb{C}$, and their decompositions. (1897) Applications to group algebras $\mathbb{C}G$.

Moore, Maschke (1898–99): Complete reducibility of matrix repres. (Dickson: *modular repres.*)

Wedderburn (1907): *Radical* of an algebra. Classified semisimple algebras over arbitrary fields.

Noether (1929): Module-theoretic viewpoint (repres. space = $\mathbb{C}G$-module), use of noncomm. techniques, left ideals, homom's, endom's, change of fields, ...

Group Repres. in a Nutshell

$$\mathbb{C}G \cong \mathbb{M}_{n_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{n_s}(\mathbb{C}) \quad \text{(Wed. Decomp.)}$$

- "Irred. repres. = simple module". The i-th simple component gives one simple module $V_i = \mathbb{C}^{n_i}$.
- Magic Equation follows by counting dimensions; $s = \#\{\text{conj. classes}\}$ follows from counting *center* dimensions.

Influx From Number Th. and Arithmetic

- "$\chi(1)$ divides $|G|$" used **Frobenius Integrality:**
  $|\mathrm{Cl}\,(g)| \cdot \chi(g)/\chi(1)$ is an algebraic integer.
- Burnside's $p^a q^b$-Theorem used the same.
- $\forall$ irred. χ, $\mathbb{Q}\,(\chi) := \sum_{g\in G} \mathbb{Q} \cdot \chi(g)$ is an **algebraic number field**, due to a product formula

$$\chi(h)\,\chi(k) = \frac{\chi(1)}{|G|} \sum_{g\in G} \chi(h\,k^g).$$

The "char. field" $\mathbb{Q}\,(\chi)$ is an *abelian* Galois ext./$\mathbb{Q}$.

- The repres. D with char. χ may not be defined over $\mathbb{Q}\,(\chi)$. "How far up" you have to go to "define" D leads to the theory of **Schur Index** of repres's, and later to **Brauer's Splitting Field Thm**.
- $\forall$ irred. $\chi = \chi_D$, the **Frobenius-Schur indicator**:

$$s\,(\chi) = |G|^{-1} \textstyle\sum_{g\in G} \chi(g^2) \in \{0,\ \pm 1\}.$$

It is 1 if D is equiv. to a *real repres.*, -1 if this is not so but χ is *real-valued*, and 0 if $\chi(G) \not\subseteq \mathbb{R}$.

- **Involution Count**: Involut'n = element of order 2.

(Frob.-Schur) $\#\,(\text{involutions}) = \sum_\chi s\,(\chi)\,\chi(1)$.

"The Classif'n of Finite Simple Groups Is Complete." .... D. Gorenstein, 1980s

Infinite Families $\mathbb{Z}_p$, A_n $(n \geq 5)$, various families of simple groups of Lie type ($\mathrm{PSL}_n(q)$, etc.)

26 Sporadic Simple Groups: 5 Mathieu gps M_i, J_1 (1st Janko group), J_2, ... , F_2, F_1 (Monster).

Burnside (1852–1927): Pioneer
Brauer (1901–1977): Visionary
Feit-Thompson ('30–'04, '32–): Young Hotshots
Gorenstein (1923–1992): Grand Marshall

In studying (nonabel.) simple groups G, Feit-Thompson **validates** the following two approaches:

- *Look at the structure of a 2-Sylow group of G.*
- **(Brauer's Vision)** *Look at $C_G(u)$ (centralizer of an involution $u \in G$).*

Brauer-Fowler Th. (1955). *Suppose G has an "invol'n-cent'zer" $|C_G(u)| \leq m$. Then $|G| < (m^2)!$.*

It takes less than a page to prove this from the Frob-Schur formula for counting involutions.

$\exists\ |C_G(u)| = 2 \Longrightarrow G \cong \mathbb{Z}_2$.
$\exists\ |C_G(u)| = 4 \Longrightarrow G \cong A_5$.
$\exists\ C_G(u) \cong$ dihed. order 8 $\Longrightarrow G \cong A_6$ or $\mathrm{PSL}_2(7)$.

Let $q \equiv 3 \pmod 4$. **Brauer characterized** $\mathrm{PSL}_3(q)$.

$$G \cong \mathrm{PSL}_3(q) \iff \exists\, C_G(u) \cong \mathrm{GL}_2(q),$$

except when $q = 3$, in which case possibly $G \cong M_{11}$.

Opening the Floodgate — After 100 Years
Janko Group J_1 (1965)

"Janko group is a group of commercial real estate companies focused on building value through real estate ... "

JANKO | GROUP

$$|J_1| = \$\,175{,}560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19.$$

(With Hindsight) A simple gp. G is a **Ree group** iff its 2-Sylow is abelian and $\exists\, C_G(u) \cong \mathbb{Z}_2 \times \mathrm{PSL}_2(q)$ for some $q > 5$. (Actually, $q = 3^{\mathrm{odd}}$.)

What if $q = 4, 5$*?* Note $\mathrm{PSL}_2(4) \cong \mathrm{PSL}_2(5) \cong A_5$.

(Janko) There is a unique simple group J_1 **with 2-Sylows abelian, and with** $C_G(u) \cong \mathbb{Z}_2 \times A_5$**. The group is constructed from a modular repres.** $J_1 \hookrightarrow \mathrm{GL}_7(11)$.

Character Table of the Monster

$$|\text{Monster}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot \cdots \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8 \cdot 10^{53}.$$

	1	g_2	$\cdots$	g_{194}
χ_1	1	1	$\cdots$	1
χ_2	196,883	$\cdots$	$\cdots$	$\cdots$
χ_3	$\cdots$	$\cdots$	$\cdots$	$\cdots$
χ_4	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
χ_{194}	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where $196{,}883 = 47 \cdot 59 \cdot 71.$

McKay's Observation: $196{,}884 = 1 + 196{,}883\,.$

This led to Monstrous Moonshine .... and Prof. Borcherds.

We should be grateful for the character table for S_3**:**

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

But then there is always an easier one:

	1
1	1