Introductory Workshop on Combinatorial Representation Theory

Combinatorics of Lie Type - Arun Ram - 1/22/08

This is the first in a series of 3 lectures on the subject. The lecture topics will be as follows:

Lecture 1. (a) Symmetric Functions, (b) Hecke Algebras & (c) Macdonald Polynomials

Lecture 2. - (a) Groups of Lie Type (b) Loop Groups & (c) Flag Varieties
(here we'll get a special look at affinization)

Lecture 3. - (a) Tableaux (b) Path Models & (c) Crystals

Reflection Groups (over $\mathcal{O} = \mathbb{Z}$ OR $\mathbb{Z}_p$ OR $\mathbb{Z}[\xi]$)
Let $\mathbb{F}$ be the fraction field of $\mathcal{O}$ and suppose that $\mathbb{F} \subseteq \mathbb{C}$

Definition - Let $\mathfrak{h}$ be a vector space, then a reflection is a linear transformation $s \in GL(\mathfrak{h})$ s.t. $rk(1-s) = 1$.

Remark - Morally, this means that we want s to be conjugate to a matrix of the form

$$\begin{bmatrix} \xi & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{bmatrix} \text{ in } GL(\mathfrak{h}_{\mathbb{F}})$$

Definition - A reflection group is a pair $(\mathfrak{h}_{\mathcal{O}}, W_0)$ where $\mathfrak{h}_{\mathcal{O}}$ is a free $\mathcal{O}$-module and W_0 is a subgroup of $GL(\mathfrak{h}_{\mathcal{O}})$ that is generated by reflections.

Recall, the affine Weyl group is the semidirect product $W = \mathfrak{h}_O \rtimes W_0 = \{t_{\lambda^\vee} w | \lambda^\vee \in \mathfrak{h}_O, w \in W_0\}$ where $t_{\lambda^\vee} t_{\mu^\vee} = t_{\lambda^\vee + \mu^\vee}$ and $w t_{\lambda^\vee} = t_{w\lambda^\vee}$. ($W_0$ is the finite Weyl group)

If we take $\mathfrak{h} = \text{span}\{y_1, ..., y_n\}$, then recall $S(\mathfrak{h}^*) = \mathbb{C}[x_1, ..., x_n]$ and W_0 acts on these polynomials. We then get the symmetric functions $\mathbb{C}[x_1, ...x_n]^{W_0} = \{f \in \mathbb{C}[x_1, ..., x_n] | wf = f \ \forall w \in W_0\}$.

Theorem 1 (Shephard-Todd-Chevalley)
Assume W_0 is finite, then the pair $(\mathfrak{h}, W_0)$ is a reflection group iff
$\exists\ p_1, ..., p_n$ homogenous polynomials s.t. $\mathbb{C}[x_1, ..., x_n]^{W_0} = \mathbb{C}[p_1, ..., p_n]$.
(i.e. the W_0 invariant polynomials, can actually be written as a polynomial ring over algebraically independent homogeneous polynomials.)

This is, in fact, a very rigid structure which causes much of the cohomology to vanish.

Examples

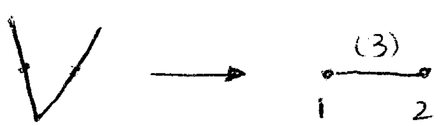
Type GL_n - Recall in this case the Weyl group is $W_0 = S_n$ and it acts on $\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{y_1, ..., y_n\}$ by permuting $y_1, ..., y_n$.

Type SL_3 - Here the Weyl group is $W_0 =< s_1, s_2 | s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 >$.
Our lattice is $\mathfrak{h}\mathbb{Z} = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \alpha_2^\vee\}$
We can draw a nice diagram of the lattice, the hyperplane arrangements, and the way in which the Weyl group permutes the chambers formed by the hyperplanes (see Figure 1 attached).

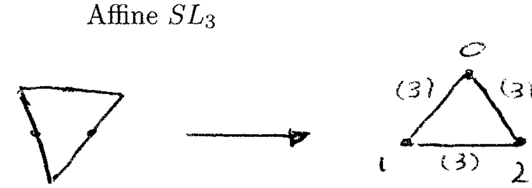
Now, we will change gears and discuss the affine Weyl group. Recall $W = \{t_{\lambda^\vee} w | \lambda^\vee \in \mathfrak{h}_{\mathcal{O}}, w \in W_0\}$. So every element is indexed by an element of the finite Weyl group (w) and an element in the lattice ($t_{\lambda^\vee}$). Hence, in our nice case, (see Figure 2) we can in effect tile all of affine space with copies of the fundamental chamber by allowing the finite Weyl group to act (with its normal reflections) and by allowing shifts by the lattice points. (We get a copy of the finite Weyl group centered at each lattice point)

In these lucky cases, (i.e. SL_3 and affine SL_3) the Dynkin diagram is actually the dual graph (I, E) of the fundamental chamber. (see Diagrams below)

SL_3



Affine SL_3



Furthermore, by adding the labels to the edges of the Dynkin diagrams we can keep track of the angles. This extra information also tells us about the relations in W_0. Namely, if we label the edge connecting i and j by the number m_{ij} then we get the following presentation for W_0.
$W_0 =< s_i \mid s_i^2 = 1,\quad s_i s_j s_i ... = s_j s_i s_j ... >$ where each product has exactly m_{ij} terms.

The DAHA: The Double Affine Hecke Algebra

Let $X = \{q^k X^\mu | k \in \mathbb{Z}, \mu \in \mathfrak{h}_{\mathbb{Z}}^*\}$ with $qX^\mu = X^\mu q$ and $X^\mu X^\nu = X^{\mu+\nu}$.
Now the affine Weyl group W acts on X via $wX^\mu = X^{w\mu}$, $wq = q$, and $t_{\lambda^\vee} X^\mu = q^{-<\lambda^\vee, \mu>} X^\mu$
If we fix parameters $t_i^{1/2}$ for $i = 1, ..., n$ in the base field (so they commute with all of the generators) the we can define the DAHA ($\tilde{H}$) as follows:
$\tilde{H}$ is generated by $T_0, ...T_n, X$ where $T_i^2 = (t_i^{1/2} - t_i^{-1/2})T_i + 1$
and $T_i T_j T_i ... = T_j T_i T_j ...$ when each of these products has m_{ij} factors.

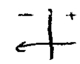
Now define $X^{s_i \mu} = T_i X^\mu T_i^{-1}$ if $< \mu, \alpha_i^\vee >= 0$ and $X^{s_i \mu} = T_i X^\mu T_i$ if $< \mu, \alpha_i^\vee >= 1$
(Recall, the inner product $< \lambda^\vee, \mu >= \mu(\lambda^\vee)$ comes from the pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$. In this case, it is measuring the distance from μ to the hyperplane h^{α_i}.)

Dunkl-Cherednik Operators

We can form a specific basis for $\tilde{H}$ assuming that the parameters $\{t_i\}$ are in $\mathbb{C}$ already. Then $\tilde{H} = \mathbb{C}[q^{\pm 1}]\text{-span}\{X^\mu T_w | \mu \in \mathfrak{h}_{\mathbb{Z}}^*,\ w \in W\}$ where $T_w = T_{i_1} ... T_{i_l}$ whenever $w = s_{i_1} ... s_{i_l}$ is a minimal length walk to w.

Let's now contrast the above work with the (Single) Affine Hecke Algebra (AHA).
$H = \text{span}\{T_w | w \in W\}$ We can actually write down another presentation,
$H = \text{span}\{Y^{\lambda^\vee} T_w | w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$.

Then $Y^{\lambda^\vee} = T_{i_1}^{\epsilon_1} ... T_{i_l}^{\epsilon_l}$ where $\epsilon_k = \begin{cases} 1 \text{ if the } k\text{th step is} \\ \\ -1 \text{ if the } k\text{th step is} \end{cases}$



Signs in the steps come from orienting all of the hyperplanes so that the fundamental chamber gets all +'s and then extending so that all parallel hyperplanes get the same orientation (see Figure 3). Then notice we get that $Y^{\lambda^\vee} Y^{\mu^\vee} = Y^{\lambda^\vee + \mu^\vee} = Y^{\mu^\vee} Y^{\lambda^\vee}$, they commute!

So we now have $\tilde{H} = \mathbb{C}[q^{\pm 1}]\text{-span}\{X^\mu T_w | \mu \in \mathfrak{h}^*_{\mathbb{Z}},\ w \in W\}$ and contained inside $\tilde{H}$ we have $H = \text{span}\{Y^{\lambda^\vee} T_w | w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}\}$.

Definition - the polynomial representation (denoted $\mathbf{1}$) is defined by the action $T_i \mathbf{1} = t_i^{1/2} \mathbf{1}$ for $i = 0, ..., n$.

So notice that, $\tilde{H}\mathbf{1} = \mathbb{C}[q^{\pm 1}]\text{-span}\{X^\mu \mathbf{1} | \mu \in \mathfrak{h}_{\mathbb{Z}}\}$. They look like polynomials! Can we find the eigenvalues?

The Dunkl-Cherednik Operators (the $Y^{\lambda^\vee}$ as operators on X^μ) are actually the integrals for the system of differential equations above.

Macdonald Polynomials

Definition - The non-symmetric Macdonald polynomials are E_μ, the eigenvectors of $Y^{\lambda^\vee}$ acting on $\tilde{H}\mathbf{1}$.

There are symmetric versions of these polynomials also.
Let $\mathbf{1}_0 \tilde{H} \mathbf{1} = \{p\mathbf{1} | wp = p \text{ for } w \in W_0\}$ (these are all symmetric polynomials).
Then take $m_{\lambda^\vee} = \sum_{\gamma^\vee \in W_0 \lambda^\vee} Y^{\gamma^\vee}$. These are the monomial symmetric functions.

Definition - the Macdonald polynomials are $P_\mu(q, t_1, ..., t_n)$ (or just P_μ), the eigenvectors of $m_{\lambda^\vee}$ acting on $\mathbf{1}_0 \tilde{H} \mathbf{1}$.

Now, recall that eigenvectors are only unique up to scaling. Hence, if we want unique polynomials we need to normalize them.
Take $E_\mu = X^\mu +$ lower terms and $P_\mu = X^\mu +$ lower terms

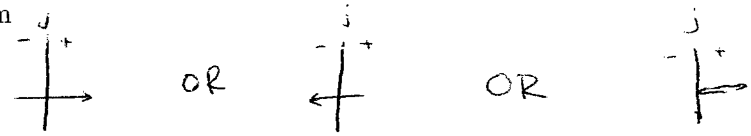
Once we have these two sets of polynomials, some natural questions arise:
QUESTIONS
(1) Can we expand the E_μ in terms of X^ν?
(2) Can we expand the P_μ in terms of m_ν?
(3) Can we expand the P_μ in terms of the Schur functions s_ν?

In type GL_n these questions have been answered by recent work: (1) by HHL II , (2) by HHL I, (3) by Assaf.

Hall-Littlewood Polynomials

These appear in the special case of $P_\mu(q, t_1, ..., t_n)$ when we take $q = 0$ and $t = t_1 = t_2 = ... = t_n$.

Definition - A positively folded alcove walk is a sequence of steps, where a step of type j is a step of the form



where j labels the family of hyperplane that is being crossed.

Let $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$, and let $t_{\lambda^\vee} = s_{i_1}...s_{i_l}$ be a reduced word. Then there is a nice result due to C. Schwer.

Theorem 2 *(C. Schwer)*

$$P_\lambda(0,t) = \sum_{p \in B_t(\lambda)} (1-t)^{f(p)} t^{\frac{1}{2}[l(i(p))+l(\phi(p))-f(p)]} X^{wt(p)}$$

where $B_t(\lambda) = \{$positively folded alcove walks of type $i_1, ..., i_l$ beginning in the 0-hexagon$\}$
$i(p)$ is the initial position of p, $\phi(p)$ is the final position of p,
$f(p)$ is the number of folds in p, and $wt(p)$ (the 'weight' of p) is the final hexagon of p

Closing Remark: The Littelmann paths are the elements of the set $B(\lambda) = \{p \in B_t(\lambda) | l(i(p)) + l(\phi(p)) - f(p) = 0\}$. These paths form a nice crystal. You can also see $B(\lambda)$ appearing with Schur functions as $s_\lambda = \sum_{p \in B(\lambda)} X^{wt(p)}$.

Combinatorics of Lie Type

Lecture 1: Symmetric functions

Hecke algebras

Macdonald Polynomials

Lecture 2: Groups of Lie Type

Loop Groups

Flag varieties

Lecture 3: Tableaux

Path models

Crystals

Reflection groups $\mathcal{O} = \mathbb{Z}$ or $\mathbb{Z}_p$ or $\mathbb{Z}[\xi]$

$\mathbb{F}$ = fractions of $\mathcal{O}$, $\mathbb{C} \supseteq \mathbb{F}$ a field.

A reflection is $s \in GL(\mathfrak{h})$ such that $rk(1-s) = 1$

Morally

s is conjugate to $\begin{pmatrix} \xi & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ in $GL(\mathfrak{h}_{\overline{\mathbb{F}}})$

A reflection group is a pair $(\mathfrak{h}_{\mathcal{O}}, W_0)$

$\mathfrak{h}_{\mathcal{O}}$ is a free $\mathcal{O}$-module

W_0 is a subgroup of $GL(\mathfrak{h}_{\mathcal{O}})$

generated by reflections. The affine Weyl group

$$W = \mathfrak{h}_{\mathcal{O}} \rtimes W_0 = \{ t_{\lambda^\vee} w \mid \lambda^\vee \in \mathfrak{h}_{\mathcal{O}}, w \in W_0 \}$$

$t_{\lambda^\vee} t_{\mu^\vee} = t_{\lambda^\vee + \mu^\vee}$ and $w t_{\lambda^\vee} = t_{w\lambda^\vee} w$.

W_0 acts on polynomials on $\mathfrak{h}$

$\mathfrak{h} = \mathrm{span}\{y_1, \ldots, y_n\}$, $S(\mathfrak{h}^*) = \mathbb{C}[x_1, \ldots, x_n]$

$\mathbb{C}[x_1, \ldots, x_n]^{W_0} = \{ f \in \mathbb{C}[x_1, \ldots, x_n] \mid wf = f \text{ for } w \in W_0$

Shephard-Todd-Chevalley If W_0 is finite then $(W_0, \ldots$ is a reflection group if and only if there exist homogeneous polynomials $p_1, \ldots, p_n$ such that

$$\mathbb{C}[x_1, \ldots, x_n]^{W_0} = \mathbb{C}[p_1, \ldots, p_n]$$

Examples

Type GL_n By permuting $y_1, \dots, y_n$

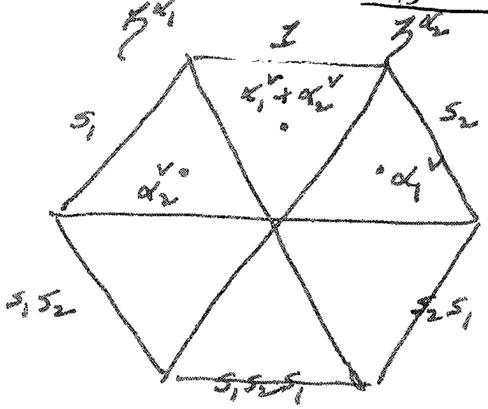
$W_0 = S_n$ acts on $\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{y_1, \dots, y_n\}$

$\mathbb{C}[x_1, \dots, x_n]^{W_0} = \mathbb{C}[p_1, \dots, p_n]$ with $p_k = x_1^k + \dots + x_n^k$

Type SL_3 $W_0 = S_3 = \langle s_1, s_2 \mid s_i^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \rangle$

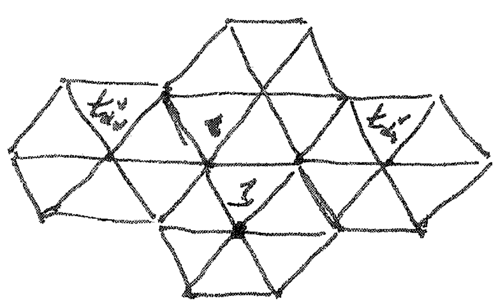
$\mathfrak{h}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \alpha_2^\vee\}$

Figure 1



has fundamental chamber 1 (in $\mathfrak{h}_{\mathbb{R}}$).

$W = \mathfrak{h}_{\mathbb{Z}} \rtimes S_3 = \{t_{\lambda^\vee} w \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}},\ w \in W_0\}$

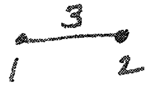


has fundamental chamber 1 (in $\mathfrak{h}_{\mathbb{R}}$).

③

DAHA - Double affine Hecke algebra $\tilde{H}$

In these lucky cases
The Dynkin diagram is the labeled dual graph (I, E) of the fundamental chamber.

gives


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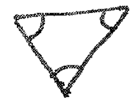
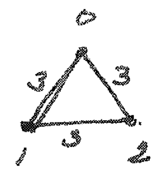
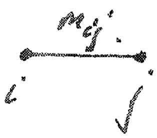

gives


and W is presented by s_i, $i \in I$, with

$$s_i^2 = 1 \quad \text{and} \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \quad \text{if}$$


Let

$$\mathcal{X} = \{ q^k X^\mu \mid k \in \mathbb{Z},\ \mu \in \mathfrak{h}^*_{\mathbb{Z}} \} \quad \text{with}$$

$$q X^\mu = X^\mu q \quad \text{and} \quad X^\mu X^\nu = X^{\mu+\nu}.$$

W acts on $\mathcal{X}$ by

$$w X^\mu = X^{w\mu} \quad \text{and} \quad t_{\lambda^\vee} X^\mu = q^{-\langle \lambda^\vee, \mu \rangle} X^\mu$$

The DAHA $\tilde{H}$ has generators $T_0, \ldots, T_n$ and X^μ

$$T_i^2 = (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}) T_i + 1, \qquad \underbrace{T_i T_j T_i \cdots}_{m_{ij}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij}}.$$

$$X^{s_i \mu} = T_i X^\mu T_i^{-1}, \quad \text{if } \langle \mu, \alpha_i^\vee \rangle = 0$$

$$X^{s_i \mu} = T_i X^\mu T_i, \quad \text{if } \langle \mu, \alpha_i^\vee \rangle = 1.$$

Dunkl-Cherednik operators

$$\tilde{H} = \mathbb{C}[q^{\pm 1}]\text{-span}\{ X^{\mu} T_w \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}},\ w \in W \}$$

where

$$T_w = T_{i_1} \cdots T_{i_\ell} \quad \text{if} \quad w = s_{i_1} \cdots s_{i_\ell} \quad \text{is a minimal}$$

length walk to w.

$$H = \text{span}\{ T_w \mid w \in W \}$$

$$= \text{span}\{ y^{\lambda^\vee} T_w \mid \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}},\ w \in W_0 \}$$

where, if $t_{\lambda^\vee} = s_{i_1} \cdots s_{i_\ell}$ then

$$y^{\lambda^\vee} = T_{i_1}^{\epsilon_1} \cdots T_{i_\ell}^{\epsilon_\ell} \quad \text{with}$$

$$\epsilon_k = \begin{cases} +1, & \text{if the } k^{\text{th}} \text{ step is } \xrightarrow{-\,|\,+} \\ -1, & \text{if the } k^{\text{th}} \text{ step is } \xleftarrow{-\,|\,+} \end{cases}$$

The <u>polynomial</u> <u>representation</u> is

$$\tilde{H}\mathbb{1} = \mathbb{C}[q^{\pm 1}]\text{-span}\{ X^{\mu}\mathbb{1} \mid \mu \in \mathfrak{h}^*_{\mathbb{Z}} \}$$

where $T_i \mathbb{1} = t_i^{\frac{1}{2}} \mathbb{1}$ for $i = 0, 1, \ldots, n$

The <u>Dunkl-Cherednik</u> <u>operators</u> are the $y^{\lambda^\vee}$ as operators on $\tilde{H}\mathbb{1}$.

Macdonald polynomials

Note: $Y^{\lambda^\vee} Y^{\mu^\vee} = Y^{\lambda^\vee + \mu^\vee} = Y^{\mu^\vee + \lambda^\vee}$.

The nonsymmetric Macdonald polynomials

E_μ are the eigenvectors of $Y^{\lambda^\vee}$ on $\tilde{H}\mathbb{1}$

Let

$$\mathbb{1}_0 \tilde{H} \mathbb{1} = \{ p\mathbb{1} \mid wp = p \text{ for } w \in W_0 \}$$

$$m_{\lambda^\vee} = \sum_{\gamma^\vee \in W_0 \lambda^\vee} Y^{\gamma^\vee} \qquad \left(\begin{array}{l}\text{monomial}\\ \text{symmetric function}\end{array}\right)$$

The Macdonald polynomials

P_μ are the eigenvectors of $m_{\lambda^\vee}$ in $\mathbb{1}_0 \tilde{H} \mathbb{1}$.

Normalization: $E_\mu = X^\mu +$ lower terms

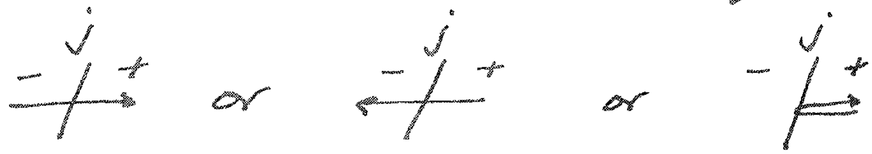
$P_\mu = X^\mu +$ lower terms

Questions:

	Type GL_n
(1) Expand E_μ in terms of X^μ	HHL II
(2) Expand P_μ in terms of m_μ	HHL I
(3) Expand P_μ in terms of s_ν	Assaf

Hall-Littlewood polynomials $P_\mu(0,t)$.

A *positively folded alcove walk* is a sequence of steps, where a *step of type j* is



Let $\mu \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$ and choose a reduced word

$$t_\mu = s_{i_1} \cdots s_{i_\ell}.$$

Then

$$P_\mu(0,t) = \sum_{p \in B_t(\mu)} (1-t)^{f(p)} t^{\frac{1}{2}(\ell(\iota(p)) + \ell(\varphi(p)) - f(p))} X^{wt(p)}$$

where

$$B_t(\mu) = \left\{ \begin{array}{l} \text{positively folded alcove walks of} \\ \text{type } i_1, \ldots, i_\ell \text{ beginning in the 0 hexagon} \end{array} \right\}$$

$\iota(p) =$ initial position of p

$wt(p) =$ final hexagon of p

$\varphi(p) =$ final position of p

$f(p) =$ (number of folds in p).

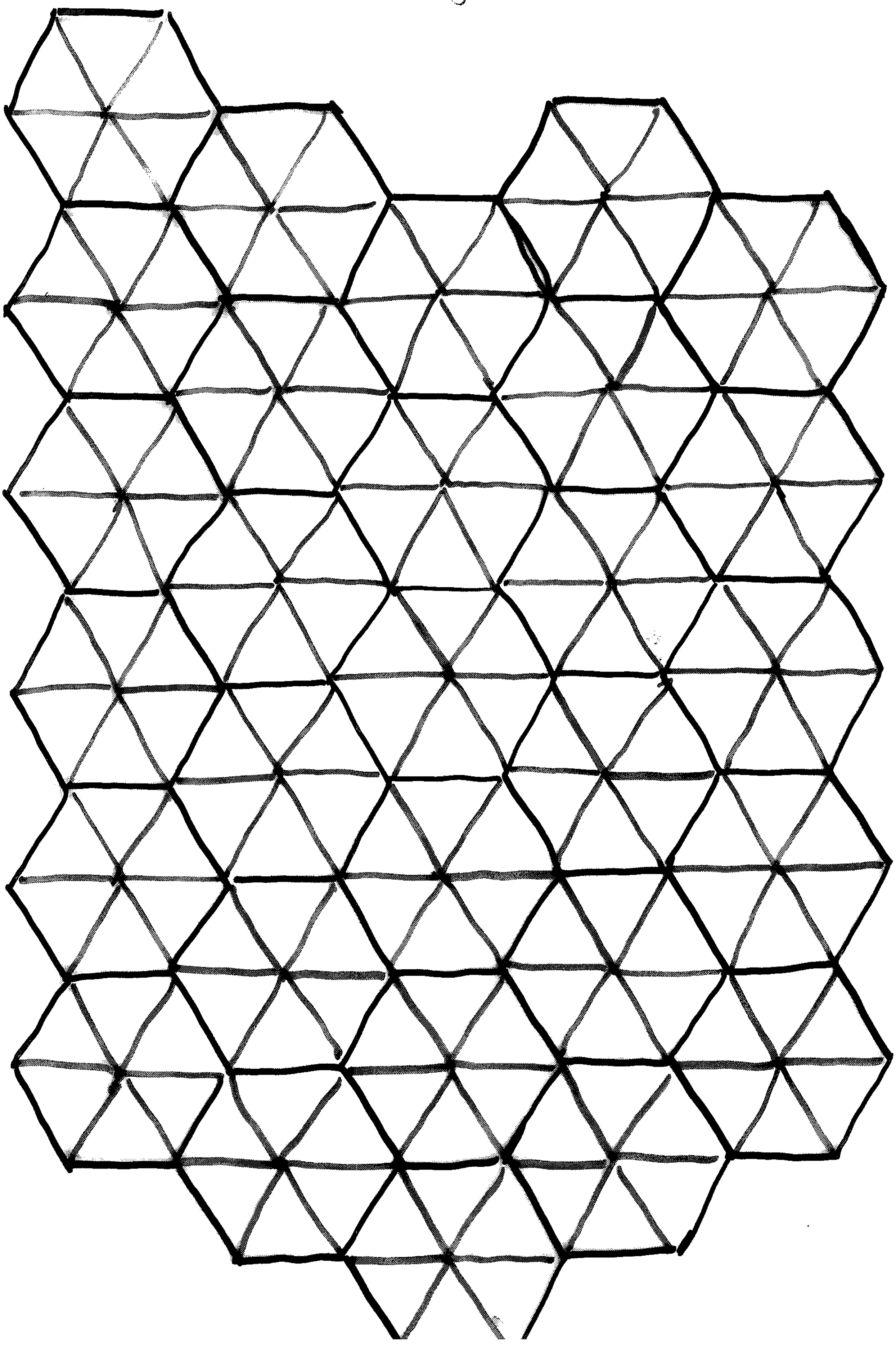
Remark The Littelmann paths (LS walks) are

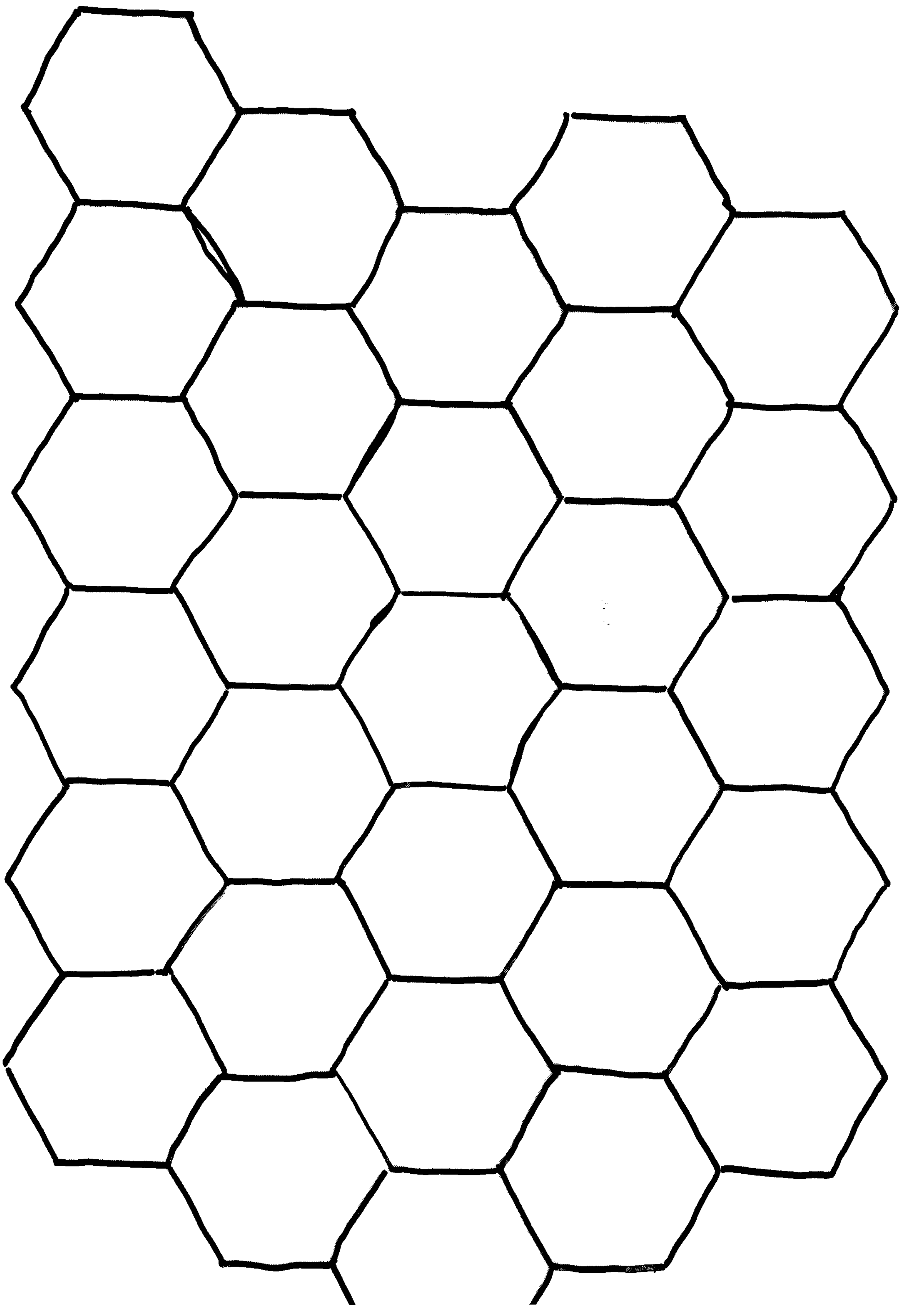
$$B(\mu) = \{ p \in B_t(\mu) \mid \ell(\iota(p)) + \ell(\varphi(p)) - f(p) = 0 \}$$

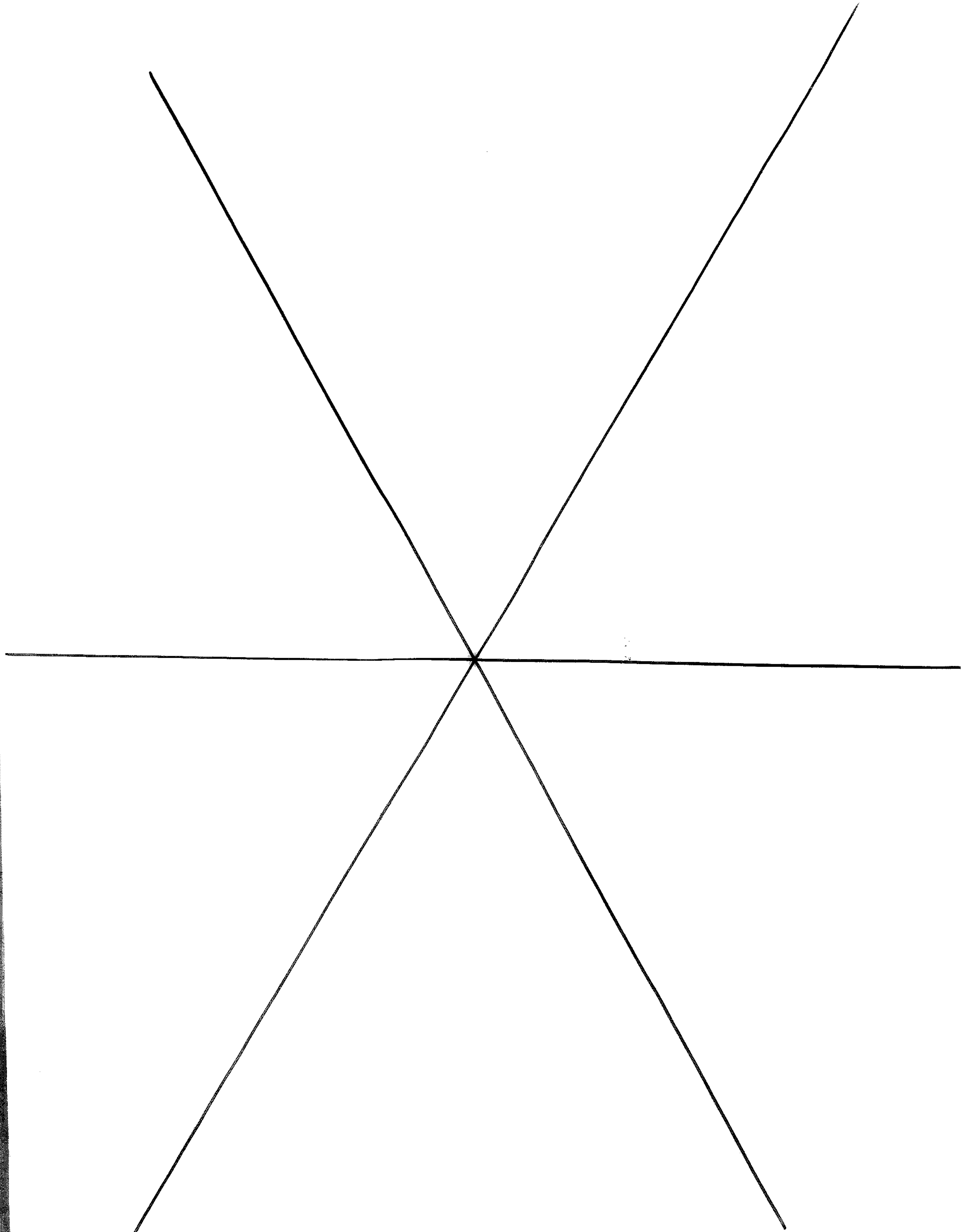
and

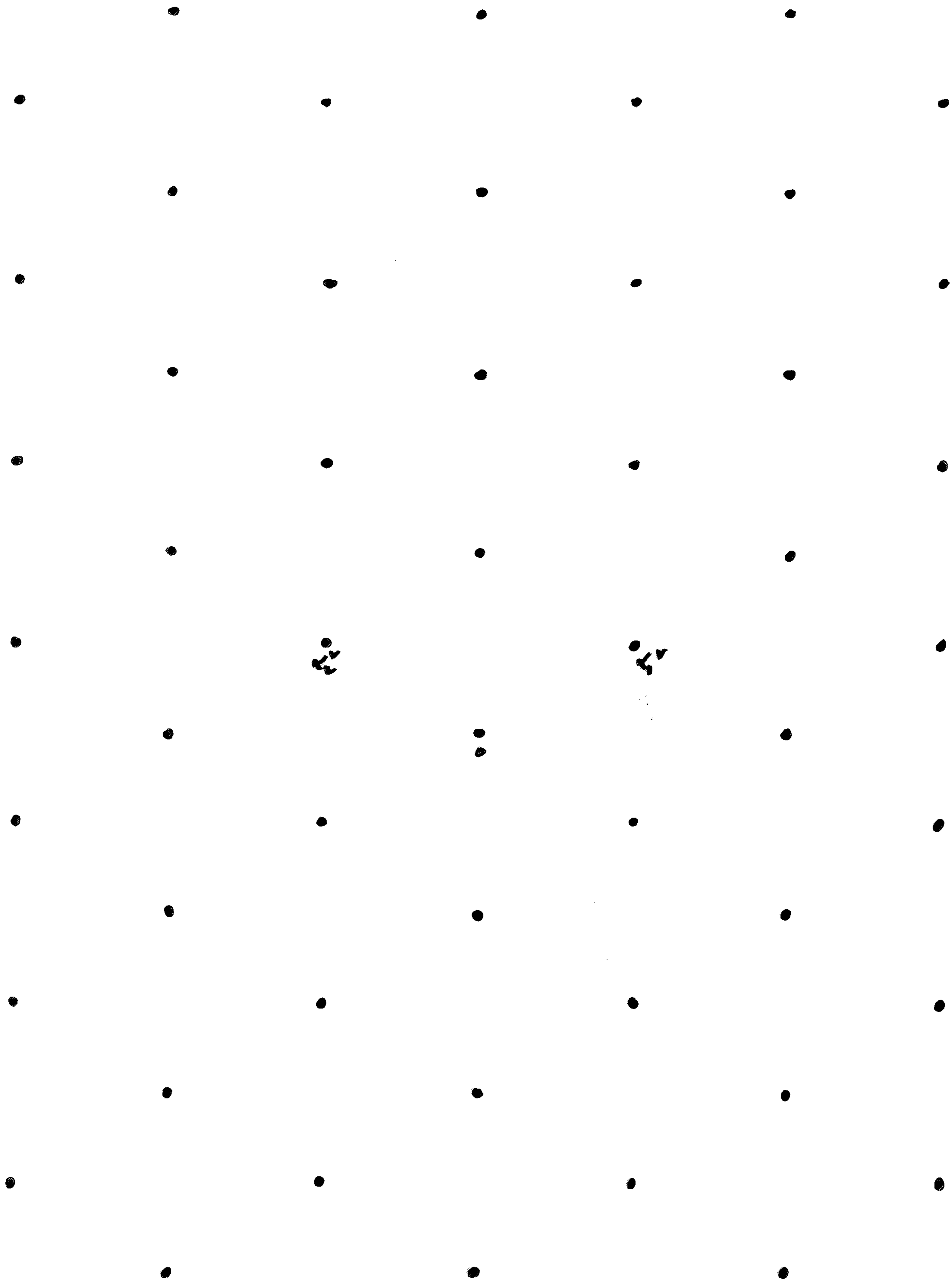
$$s_\mu = \sum_{p \in B(\mu)} X^{wt(p)}.$$

Figure 2

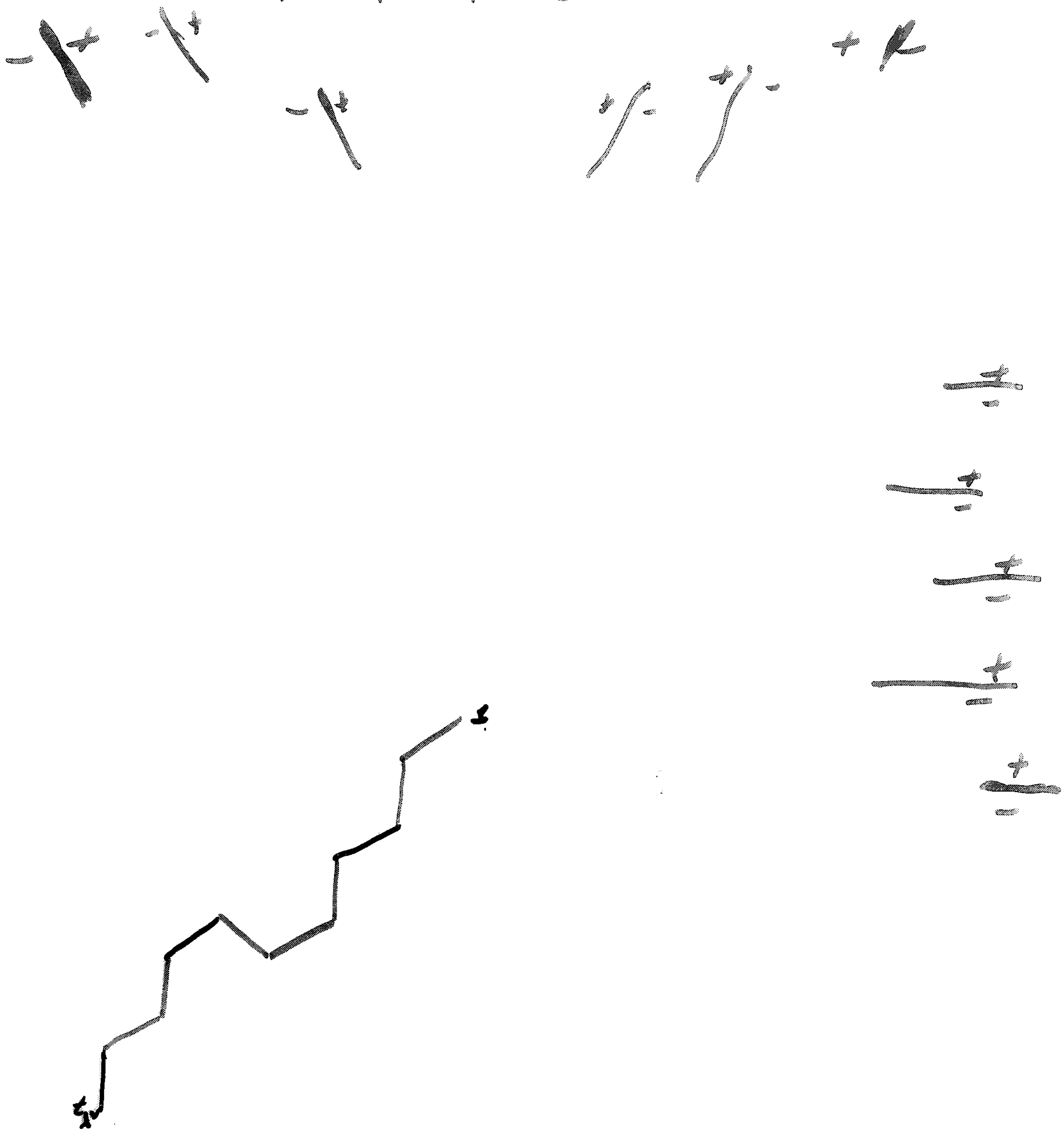








$t_{2v} = s_1 s_2 s_0 s_1 s_2 s_1 s_0 s_1 s_2 s_0$



— = 1

— = 2

— = 0