Complex reflection groups in representations of finite ~~reflective~~ reductive groups

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January 2008

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- A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

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Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let $S(V)$ denote the symmetric algebra of V, isomorphic to the polynomial ring $K[X_1, X_2, \ldots, X_r]$.
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Example

For $G = \mathfrak{S}_r$, one may choose

$$\begin{cases} f_1 = X_1 + \cdots + X_r \\ f_2 = X_1X_2 + X_1X_3 + \cdots + X_{r-1}X_r \\ \vdots \qquad \vdots \\ f_r = X_1X_2 \cdots X_r \end{cases}$$

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$$G(2, 2, r) = W(\mathrm{D}_r)$$
$$G_{23} = H_3 \ , \ \ G_{28} = F_4 \ , \ \ G_{30} = H_4$$
$$G_{35,36,37} = E_{6,7,8} \, .$$

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$$\mathrm{U}_n = (X = Y = \mathbb{Z}^n, R = R^\vee = A_n\, ;\, \phi = -1)$$

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = \frac{\varepsilon_{\mathbb{G}} x^N}{\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det_V(1 - xw\phi)}} = x^N \prod_d \Phi_d(x)^{a(d)}$$

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- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

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Cauchy theorem

The (polynomial) order of an admissible subgroup divides the (polynomial) order of the group.

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3. The (polynomial) index of the normalizer in G of a Sylow $\Phi(x)$–subgroup is congruent to 1 modulo $\Phi(x)$.

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For each d $(1 \leq d \leq n)$, $\mathrm{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{[\frac{n}{d}]}$

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Assume $n = md + r$ with $r < d$. Then a minimal d–split Levi subgroup has shape $\mathrm{GL}_1(q^d)^m \times \mathrm{GL}_r(q)$.

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- Then the Sylow ℓ–subgroups of G are nothing but the Sylow ℓ–subgroups S_ℓ of $S = \mathbf{S}^F$ ($\mathbf{S}$ a Sylow $\Phi_d(x)$–subgroup of $\mathbf{G}$).

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- We have

$$N_G(S_\ell) = N_G(\mathbf{S}) \text{ and } C_G(S_\ell) = C_G(\mathbf{S}) .$$

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

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$$N_G(\mathbf{L})/L \simeq N_G(\mathbf{S})/C_G(\mathbf{S}) \simeq N_W(\mathbb{L})/W'$$

(where W' is the Weyl group of $\mathbf{L}$).

Denote that group by $W_{\mathbb{G}}(\mathbb{L})$.

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- The "number of Sylow congruence" translates to

For ζ a primitive d-th root of the unity, we have

$$|W_{\mathbb{G}}(\mathbb{L})| = \mathbb{G}(\zeta)/\mathbb{L}(\zeta)\,.$$

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For $n = mr + d$ $(d < r)$, we have $W_{\mathbb{G}}(\mathbb{L}) \simeq C_d \wr \mathfrak{S}_r$

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The group $W_{\mathbb{G}}(\mathbb{L})$ is called the d–cyclotomic Weyl group.

If G is split, the 1–cyclotomic Weyl group is nothing but the ordinary Weyl group W.

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- Generic degree : For $\gamma \in \mathrm{Un}(\mathbb{G})$ there is $\mathrm{Deg}_\gamma(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\gamma(x)_{|x=q} = \gamma_q(1)\,.$$

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$$\mathrm{Deg}_\lambda(x) = \frac{(x-1)\cdots(x^n-1)\prod_{j>i}(x^{\beta_j} - x^{\beta_i})}{x^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod_i \prod_{j=1}^{\beta_i}(x^j-1)}$$

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  Note. The polynomial $\dfrac{|\mathbb{G}|(x)}{\mathrm{Deg}_\gamma(x)}$ belongs to $\mathbb{Z}[x]$ and is called the (generic) Schur element of γ.

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Theorem

For any generic Levi subgroup $\mathbb{L}$ of $\mathbb{G}$, there exist adjoint linear maps

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Un}(\mathbb{L}) \longrightarrow \mathbb{Z}\mathrm{Un}(\mathbb{G}) \quad \text{and} \quad {}^*R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Un}(\mathbb{G}) \longrightarrow \mathbb{Z}\mathrm{Un}(\mathbb{L})\,.$$

which specialize to Deligne–Lusztig maps for $x = q$.

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- A binary relation on $\mathcal{S}_d(\mathbb{G})$ –

Definition :

$$(\mathbb{M}_1, \mu_1) \leq (\mathbb{M}_2, \mu_2)$$

if and only if μ_2 occurs in $R_{\mathbb{M}_1}^{\mathbb{M}_2}(\mu_1)$.

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   - For $(\mathbb{L}, \lambda)$ d–cuspidal, define

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3. The sets $\mathrm{Un}(\mathbb{G}, (\mathbb{L}, \lambda))$, where $(\mathbb{L}, \lambda)$ runs over a system of representatives of the W–conjugacy classes of d–cuspidal pairs, form a partition of $\mathrm{Un}(\mathbb{G})$.

For $(\mathbb{L}, \lambda)$ a *d*–cuspidal pair, we set

$$W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_W(\mathbb{L}, \lambda)/W_{\mathbb{L}} = N_G(\mathbf{L}, \lambda_q)/L \,.$$

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In the case where $\mathbb{L}$ is a minimal d–split Levi subtype, and λ is the trivial character, the above theorem specializes onto Springer–Lehrer theorem.

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There exists a collection of isometries

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1. The following diagram commute :

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\mathbb{Z}\mathsf{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda) & \xrightarrow{I_{(\mathbb{L},\lambda)}^{\mathbb{G}}} & \mathbb{Z}\mathsf{Un}(\mathbb{G},(\mathbb{L},\lambda)) \\
{\scriptstyle \mathsf{Ind}_{W_{\mathbb{M}}(\mathbb{L},\lambda)}^{W_{\mathbb{G}}(\mathbb{L},\lambda)}}\Big\uparrow & & \Big\uparrow{\scriptstyle R_{\mathbb{M}}^{\mathbb{G}}} \\
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2. For all $\chi \in \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))$, let $\gamma_\chi := \varepsilon_\chi I_{(\mathbb{L},\lambda)}^{\mathbb{G}}(\chi)$. Then if ζ is a primitive d-th root of unity, we have

$$Deg_{\gamma_\chi}(\zeta) = \varepsilon_\chi \chi(1)\,.$$