Introductory Workshop on Combinatorial Representation Theory

Characters of Symmetric Groups #2
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Fock Space Formalism (See reference book - Solitons)
Let $V = \mathbb{C}^\infty$ with basis $\{e_k\}$ for $k \in \mathbb{Z} + \frac{1}{2}$. Then $\mathfrak{gl}(V)$ acts on $\bigwedge^{\frac{\infty}{2}} V$ (the half infinite exterior power).
We know also that $K = \{k_i\}\Delta(\mathbb{Z} + \frac{1}{2})_{<0}$ is finite.
Let $V_\lambda = \bigwedge e_{\lambda_i - i + \frac{1}{2}} \in \bigwedge^{\frac{\infty}{2}} V$, we can represent such an element using a Maya Diagram (see Figure 1 attached).

If we restrict our view of $\mathfrak{gl}(V)$ to only matrices which have finitely many non-zero diagonals above the main diagonal (no restriction below) then we can define an action on $\bigwedge^{\frac{\infty}{2}} V$. So how do we do this?

For matrices with zeroes on the main diagonal we can just use our normal definition. e.g. if α_{-m} denotes the matrix with ones on the mth diagonal (above the main one) and zeroes elsewhere then our definition gives us that $\alpha_{-m} \cdot e_k = e_{k+m}$.

So then, how do we apply α_{-1} to V_{λ_∞}?
$a_{-1} \cdot V_\lambda = \alpha_{-1} \bigwedge_i e_{\lambda_i - i + \frac{1}{2}} = \sum_{k=1}^{\infty} e_{\lambda_1 - \frac{1}{2}} \wedge \cdots \wedge e_{\lambda_k - k + \frac{3}{2}} \wedge \cdots$. In this case, each 'particle' wants to move up one spot (see Figure 2 attached). If there is already a particle at that spot then the term will be zero (because of the wedge product), but otherwise moving the particle adds a box to the tableaux at the new location. Hence, $\alpha_{-1} \cdot V_\lambda = \sum_{\nu = \lambda + \square} V_\nu$.

So what if we want to act by α_{-5} instead? In this case, the action moves particles 5 spaces and (if it doesn't result in zero) then it adds a rim hook from its starting location to the new box at its ending location. (Again, see Figure 2) Hence $\alpha_{-5} \cdot V_\lambda = \sum_{\nu = \lambda + \square k} (-1)^{ht-1} V_\nu$ where $\square k$ denotes a rim hook of size k and ht is the height of that chosen rim hook.
Notice this is exactly the same as the Murnaghan-Nakayama formula $P_k S_\lambda = \sum_{\nu = \lambda + \square k} (-1)^{ht-1} S_\nu$

Remark: It turns out the that coefficient of V_λ in $\prod \alpha_{-\mu_i} V_\emptyset$ is exactly equal to the coefficient of S_λ in P_μ, namely, χ^λ_μ.

Further Remark: If we instead make our asis $\{e_k\}$ orthonormal, then the V_λ turn out to be orthonormal also, and we can express χ^λ_μ as an inner product $(V_\lambda, \prod \alpha_{-\mu_i} V_\emptyset)$.

Recall yesterday we looked at $n^{\underline{k}}\frac{\chi^{\lambda}_{(\mu,1,\ldots,1)}}{\dim\lambda}$ where $k = |\mu|$ and $n = |\lambda|$. Last time, we showed that this could be expressed as a polynomial in λ when μ was small and fixed.

Now we can say that $n^{\underline{k}}\frac{\chi^{\lambda}_{(\mu,1,\ldots,1)}}{\dim\lambda} = \frac{(V_\lambda, e^{\alpha_{-1}}\prod \alpha_{-\mu_i}V_\emptyset)}{(V_\lambda, e^{\alpha_{-1}}V_\emptyset)}$. (In fact, the denominator of the right hand side is exactly $\frac{\dim\lambda}{n!}$

Remark: We use the exponential notation above for simplicity. For example, in the denominator, the only term that will make a non-zero contribution is $\frac{(\alpha_{-1})^n}{n!}$.

Now we need to define the action of a diagonal element such as $\mathrm{diag}(a_1, \ldots, a_k, \ldots) \in \mathfrak{gl}(V)$ on $\bigwedge^{\frac{\infty}{2}} V$. Unfortunately, this is very difficult because we can write $\mathrm{diag}(a_1, \ldots) \bigwedge\limits_i e_{k_i} = (\sum a_{k_i}) \bigwedge\limits_i e_{k_i}$, but this doesn't actually have any meaning apriori. So we must repair this by instead defining $\mathrm{diag}(a_1, \ldots) \bigwedge e_{k_i} = (\sum\limits_i a_{k_i} - \sum\limits_{k \in (\mathbb{Z}+\frac{1}{2})_{<0}} a_k) \bigwedge e_{k_i}$.

This second sum removes all of the negative half-integer terms, hence we now have a finite sum.

Recall, if we have a finite wedge and use the leibniz rule, then we can directly define a representation of a lie algebra. The rules of commutation in this case will be well-defined up to a scalar operator. Therefore, this defines a projective representation of $\mathfrak{gl}(V)$. Notice that $\alpha_m \cdot e_k = e_{k-m}$ is just a shift, hence α_m and α_n will commute in $\mathfrak{gl}(V)$. To contrast this, on $\bigwedge^{\frac{\infty}{2}} V$ we instead get that $[\alpha_m, \alpha_n] = m\delta_{m,-n}$.

Now we can define an operator P_r via $P_r e_k = k^r e_k$ (so it acts as a diagonal matrix like above, where the $a_k = k^r$. Thus $P_r V_\lambda = (\sum\limits_i (\lambda_i - i + \frac{1}{2})^r - (-i + \frac{1}{2})^r) V_\lambda$. From now on, we'll denote the summation by $P_r^*(\lambda)$.

In the previous lecture, we proved that $n^{\underline{k}} \cdot \frac{\chi^{\lambda}_{(\mu,1,\ldots)}}{\dim\lambda} \in \Lambda^*$ (in fact, they form a basis).

Now, if we denote $P_\mu^* = \prod P_{\mu_i}^*$, then we get that $P_\mu^*(\lambda) = (V_\lambda, \prod P_{\mu_i} V_\lambda) = \frac{(V_\lambda, \prod P_{\mu_i} e^{\alpha_{-1}} V_\emptyset)}{(V_\lambda, e^{\alpha_{-1}} V_\emptyset)}$ (the inner product works since the $P_{\mu_i}^*$ are diagonal in the basis $\{V_\lambda\}$).

An intermediate goal then, is to try and show that this new set of functions (the $P_\mu^*(\lambda)$) span the linear space Λ^* also. Towards this end, we'll first move the operator from one side to the other by commuting. The key fact in this process is that if for any r we bracket P_r with α_{-1} $r+1$-times then we must get zero (i.e. $[\alpha_{-1}, [\alpha_{-1}, \ldots, [\alpha_{-1}, P_r]]\ldots] = 0$)

Our big goal is to now try to express our old basis of central characters ($\{f_\mu\}$) in terms of this new basis. As it turns out, doing the opposite (expressing the P_μ^*'s in terms of the f_μ's) is much easier.

Proposition

If we write $P_k^* = \sum_\mu \rho_{k,\mu} f_\mu$, then we can express the coefficients as follows:

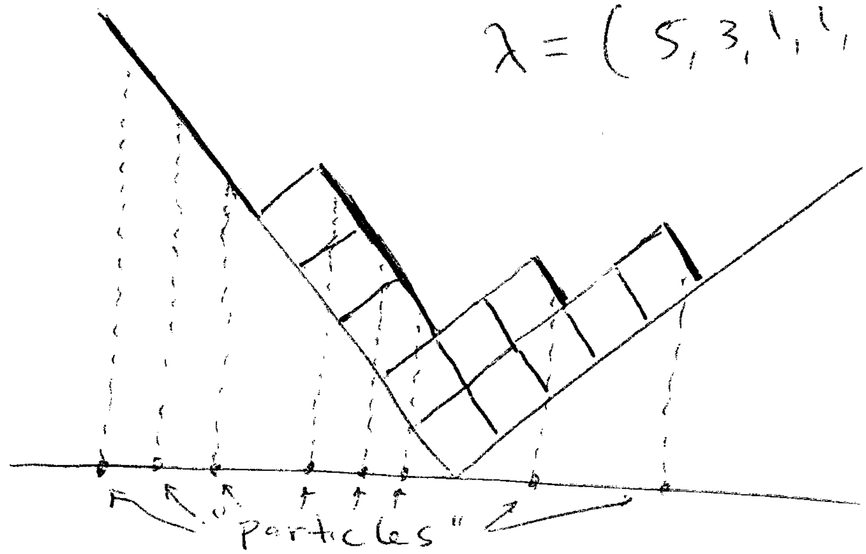
$\rho_{k,\mu} = (k-1)\frac{\prod \mu_i}{|\mu|!}[\mathcal{Z}^{k+1-|\mu|-l(\mu)}]\mathcal{S}(\mathcal{Z}^{|\mu|-1})\prod \mathcal{S}(\mu_i \mathcal{Z})$

where $\mathcal{S}(\mathcal{Z}) = \dfrac{\sinh(\frac{\mathcal{Z}}{2})}{\frac{\mathcal{Z}}{2}} = \displaystyle\sum_{k=0}^{\infty} \frac{\mathcal{Z}^{2k}}{2^{2k}(2k+1)!}$ and the brackets [...] denote picking out the coefficient of the term enclosed in the series.

Conclusion: Hence, P_k^* is the central character of some element of the form $(\bar{k}) = \frac{1}{k}C_{(k)} +$smaller terms $\in \mathbb{Q}_{\geq 0}S(n)$ where $C_{(k)}$ is a k-cycle. We call this element the completed cycle. There is also a geometric construction of the completed cycle which we'll discuss next time.

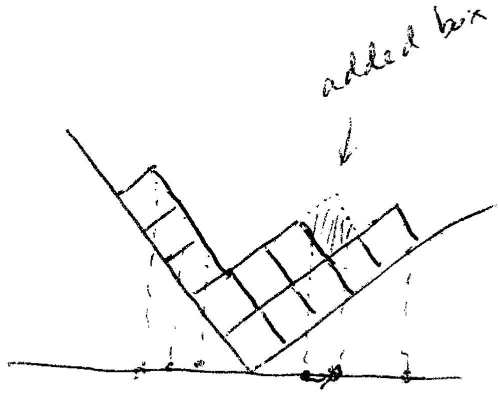
Figure 1: Maya Diagram

$\lambda = (5, 3, 1, 1, 1)$

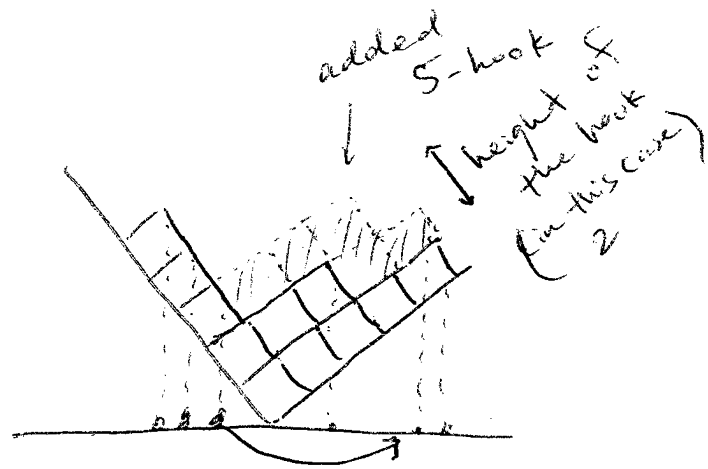


At the midpt. of each shaded edge you get a particle, there are gaps left in between.
each particle sits at the pt. $\lambda_i - i + \frac{1}{2}$

Figure 2. Action of q_m



q_{-1} moves a particle one place to the right (if not already occupied) and adds a box.



q_{-5} moves a particle 5 places (if possible) and adds a 5-hook