Introductory Workshop on Combinatorial Representation Theory

The Horn Inequalities and their Generalizations
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We'll start with a short outline for the talk
(I) - Littlewood-Richardson Numbers
(II) - Some theorems (many of which used to be conjectures until recently)
(III) - Where do Horn Inequalities come from?

Section (I) Let $\Lambda(d,n) = \{\text{partitions } \lambda = (\lambda_1 \geq ... \geq \lambda_d \geq 0...) \text{ where } \lambda_1 \leq n-d\}$. (So $\Lambda(d,\infty)$ would be partitions in which we have no bound on the first part λ_1)

(1) Representation Theory of $GL(d)$

Given $\lambda \in \Lambda(d,\infty)$ we get a corresponding irreducible representation of $GL(d)$ which we'll denote V_λ. If we consider a tensor product of two such representations, we should be able to decompose the product as a sum of other irreducibles with some multiplicities. i.e. $V_\lambda \otimes V_\mu \cong \sum_{\nu \in \Lambda(d,\infty)} C^\nu_{\lambda\mu} V_\nu$.

These numbers $C^\nu_{\lambda\mu}$ are what we call the Littlewood-Richardson Numbers.

As it turns out, the Littlewood-Richardson Numbers appear in many other places as well. For example, if we were to look instead at $(V_\lambda \otimes V_\mu \otimes V_\nu^*)^{GL(d)}$ then $C^\nu_{\lambda\mu} = \dim(V_\lambda \otimes V_\mu \otimes V_\nu^*)$.

(2) Schubert Calculus

The Littlewood-Richardson Numbers show up here as well. If we look at the Grassmannian $Gr(d,n) = \{V \subseteq \mathbb{C} | \dim(V) = d\}$, then there is a natural action of $GL(n)$ which allows us to see that $Gr(d,n) \cong GL(n)/P$ where $P = \{\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}\}$ is the parabolic subgroup of $GL(n)$ with block sizes $d \times d$, $d \times n-d$, $n-d \times d$, and $n-d \times n-d$. Inside of this parabolic, we have a Borel subgroup as well $B \subseteq P$ where B is comprised of the lower triangular matrices.

Let X_λ denote the Schubert variety (i.e. the closure of a B-orbit on $Gr(d,n)$). Notice that there are finitely many of these and that they can be indexed by the set $\Lambda(d,n)$. These are varieties, so they represent cohomology classes $[X_\lambda] \in H^*(Gr(d,n))$ called Schubert classes which we'll denote by s_λ. They actually form a $\mathbb{Z}$-basis for $H^*(Gr(d,n))$ so given any two partitions λ and μ we can write $s_\lambda s_\mu = \sum_{\nu \in \Lambda(d,n)} C^\nu_{\lambda\mu} s_\nu$, and once again, the Littlewood-Richardson Numbers appear!

We could also get them by taking an integral $C^\nu_{\lambda\mu} = \int_{Gr(d,n)} s_\lambda s_\mu s_{\nu^\vee}$.

Or by counting $C^\nu_{\lambda\mu} = \#(g_1 X_\lambda \cap g_2 X_\mu \cap g_3 X_{\nu^\vee})$ for generic elements $g_1, g_2, g_3 \in GL(n)$

Remark: One of our equivalent definitions for the Littlewood-Richardson Numbers defined them as a dimension of a variety, hence, it is clear that these numbers must be non-negative integers.

(3) Symmetric Functions

If we take $s_\lambda(x)$ to be a Schur function, then it turns out $s_\lambda(x)s_\mu(x) = \sum_{\nu\in\Lambda(\infty,\infty)} C^\nu_{\lambda\mu}s_\nu$ (they appear once again!)

We can then connect this to Representation Theory via characters, and to Schubert Calculus via degeneration. (So this begs the question, how can we connect Representation Theory to Schubert Calculus directly? That is, without going through combinatorics.)

Question: What can we say about the set $NV(d,n) = \{(\lambda,\mu,\nu) \in \Lambda(d,n)^3 | C^\nu_{\lambda\mu} > 0\}$? (NV is meant to suggest the tuples with non-vanishing Littlewood-Richardson Number)

Section (II)

Theorem 1 The Saturation Theorem (Knutson-Tao)
Let $\lambda \in \Lambda(d,\infty)$ and define $N\lambda = (N\lambda_1 \geq N\lambda_2 \geq ...)$ for any $N \in \mathbb{Z}_+$. Then $\forall N \in \mathbb{Z}_+$ we get that $C^{N\nu}_{N\lambda,N\mu} > 0$ if and only if $C^\nu_{\lambda\mu} > 0$

Remark: ($\Leftarrow$) is the easy direction, it follows from Borel-Weil (i.e. if a line bundle $L \to X$ has a $GL(d)$-invariant section, then so does $L^{\otimes N} \to X$.) The other direction is much more difficult, one way of proving it uses the hive model of $GL(d)$.

OR, we can get at it using Horn's Conjecture.
Let $\epsilon(n) = \{(\alpha,\beta,\gamma) | \exists A, B, C \in \text{Herm}_{n\times n} \text{ s.t. } A + B = C, \text{eig}(A) = \alpha, \text{eig}(B) = \beta, \text{eig}(C) = \gamma\}$ where eig(A) denotes the eigenvalues of A, etc.
The big question is, what exactly is $\epsilon(n)$? This is what Horn's Conjecture was about, and he turned out to be correct.

Theorem 2 (Klyachko)
$(\alpha,\beta,\gamma) \in \epsilon(n)$ ***if and only if***
(1) $\sum \alpha_i + \sum \beta_i = \sum \gamma_i$
(2) $\forall d < n$, $\forall(\lambda,\mu,\nu) \in NV(d,n)$ *we get* $\sum \alpha_{\tilde{\lambda}_i} + \sum \beta_{\tilde{\mu}_i} \geq \sum \gamma_{\tilde{\nu}_i}$
where $\tilde{\lambda}_i = d + 1 - i + \lambda_i$

Example (the simplest possible case)
Take $d = 1$, $\lambda = \mu = \nu = \emptyset$ (the empty partition). Then $V_\emptyset$ is the trivial representation and from the theorem we get that $\alpha_i + \beta_i \geq \gamma_i$.

Theorem 3 $(\lambda,\mu,\nu) \in NV(d,n)$ ***if and only if***
(1) $|\lambda| + |\mu| = |\nu|$
(2) $\forall r < d$, $\forall(\sigma,\rho,\tau) \in NV(r,d)$ *we get that* $\sum \lambda_{\tilde{\sigma}_i} + \sum \mu_{\tilde{\rho}_i} \geq \sum \nu_{\tilde{\tau}_i}$

Some Connections

• Saturation $\implies$ that Theorem #2 and Theorem #3 are equivalent.

Proof - Assume that $(\alpha, \beta, \gamma) \in \epsilon(n)$ are integral. Symplectic GIT equivalence implies that this is **if and only if** $\exists N > 0$ such that $(V_{N\alpha} \otimes V_{N\beta} \otimes V^*_{N\gamma})^{GL(n)} \neq 0$ which is **if and only if** $C^{N\gamma}_{N\alpha, N\beta} > 0$, and because of Saturation, this is **if and only if** $C^{\gamma}_{\alpha,\beta} > 0$.

• Theorem #3 $\implies$ Saturation.

Proof - By Theorem #3 $C^{N\nu}_{N\lambda, N\mu} > 0$ **if and only if** some inequalities hold. Similarly by Theorem #3 $C^{\nu}_{\lambda,\mu} > 0$ **if and only if** some inequalities hold. As it turns out, these inequalities are exactly the same hence we get Saturation.

Some Generalizations

• Berenstein and Sjamaar generalized Theorem #2 for all types and discussed a branching context. It was later refined by Belkale and Kumar to reduce the set of inequalities.

• Purbhoo and Sottile generalized Theorem #3 for cominuscule flag varieties. And Dierkson and Weyman also generalized Theorem #3 with representations of quivers.

• Belkale and Kumar claimed they could prove Saturation in types B and C with a factor of 2 (instead of 1).

Section (III) Now let's think about G, a reductive algebraic group over $\mathbb{C}$. Let $B \subseteq P \subseteq G$ be Borel and parabolic subgroups respectively. Then G/P is a generalized partial flag variety (if $P = B$ then it is actually a full flag variety).

Let $W^P_G = \{$minimal coset representatives of $W_G/W_P\}$. Then for $\lambda \in W^P_G$ we can associate a Schubert variety $X_\lambda = \overline{B\lambda P/P} \subseteq G/P$

Recall the Schubert classes $s_\lambda = [X_\lambda] \in H^*(G/P)$ form a nice $\mathbb{Z}$-linear basis. This brings up a natural question: When is $s_{\lambda_1} s_{\lambda_2} \cdots s_{\lambda_k} = 0$?

Let $T_\lambda =$ the tangent space at the identity (that has been translated to make the identity a smooth point). We can then write it as $T_\lambda = T_{ep}\lambda^{-1}X_\lambda \subseteq T_{ep}G/P \cong \mathfrak{g}/\mathfrak{p}$.

Proposition #1

$\prod_{i=1}^{k} s_{\lambda_i} \neq 0$ **if and only if** for general elements $p_1, ...p_k \in P$ we have that $p_1 T_{\lambda_1}, ..., p_k T_{\lambda_k}$ are transverse in $\mathfrak{g}/\mathfrak{p}$ (recall, transverse means that the sum of the codimensions is equal to the codimension of the intersection).

Remark: By considering all possible $p_1, ..., p_k \in P$ we can see that if $\prod s_{\lambda_i} \neq 0$ then we get a line bundle $L \to (P/B)^k$ with a non-zero P-invariant section. (This gives us many connections to Representation Theory).

Proposition #2

Let $\phi : \mathfrak{g}/\mathfrak{p} \twoheadrightarrow Z$. If $p_1 T_{\lambda_1}, ..., p_k T_{\lambda_k}$ are transverse in $\mathfrak{g}/\mathfrak{p}$ then $\sum \text{codim}\phi(p_i T_{\lambda_i}) \leq \dim(Z)$.

Remark: If we restrict to $\mathfrak{p}$-equivariant maps ϕ, then we actually get that $\sum \text{codim}\phi(T_{\lambda_i}) \leq \dim(Z)$, which is a nice combinatorial formula with now dependence on p_i.

Proposition #3
Let $B \subsetneq R \subsetneq P \subseteq G$ (so R is an intermediate subgroup between the chosen parabolic and the Borel). Then there is a fibration $G/R \to G/P$ with fiber P/R that sends $\lambda_i \rho_i \in W_G^R$ to $\lambda_i \in W_G^P$. If we assume that $\prod s_{\rho_i} \neq 0$, then
$\prod s_{\lambda_i} \neq 0$ **if and only if** $\prod s_{\lambda_i \rho_i} \neq 0$.

Conclusion: Putting Propostions #1, #2, and #3 together... everytime $\prod s_{\rho_i} \neq 0 \in H^*(P/R)$ we get the necessary inequalities for $\prod s_{\lambda_i} \neq 0$ and it is a fact that the Horn inequalities arise in exactly this way.

Horn Inequalities and their Generalizations.

I. Littlewood–Richardson numbers.

$$\Lambda(d,n) = \{ \text{partitions} \quad \lambda = n-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0 \}$$
$$\Lambda(d,\infty) = \{ \text{partitions} \ . \ \lambda = \lambda_1 \geq \cdots \geq \lambda_d \geq 0 \}.$$

1. Representation Theory of $GL(d)$.

$$\lambda \in \Lambda(d,\infty) \rightsquigarrow V_\lambda - GL(d) \text{ irrep.}$$

$$V_\lambda \otimes V_\mu \cong \sum_{\nu \in \Lambda(d,\infty)} c_{\lambda\mu}^{\nu} V_\nu .$$

$$c_{\lambda\mu}^{\nu} = \dim \left(V_\lambda \otimes V_\mu \otimes V_\nu^{*} \right)^{GL(d)}.$$

2. Schubert Calculus.

$$GL(n) \circlearrowright Gr(d,n) = \{ V \subseteq \mathbb{C}^n \mid \dim V = d \} \leftarrow \text{Grassmanni}$$
$$\cong GL(n)/P \qquad P = \left\{ \begin{pmatrix} \overbrace{*}^{d} & 0 \\ * & \underbrace{*}_{n-d} \end{pmatrix} \right\}$$

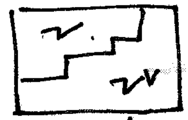
$B \subset P$ Borel subgroup.
$$B = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{pmatrix} \right\}.$$

X_λ – Schubert varieties = closures of the (finitely many) B-orbits on $Gr(d,n)$. Indexed by $\lambda \in \Lambda(d,n)$.

$S_\lambda = [X_\lambda] \in H^*(Gr(d,n)) \longleftarrow$ schubert class.
$\mathbb{Z}$-basis for $H^*(Gr(d$

$$S_\lambda S_\mu = \sum_{\nu \in \Lambda(d,n)} c_{\lambda\mu}^{\nu} S_\nu .$$

$$c_{\lambda\mu}^{\nu} = \int_{Gr(d,n)} S_\lambda S_\mu S_{\nu^\vee}.$$

d  n−d.

$$= \#\left(g_1 X_\lambda \cap g_2 X_\mu \cap g_3 X_{\nu^\vee}\right) \qquad g_1, g_2, g_3 \in G \text{ generic.}$$

Note: $c_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$.

3. Symmetric functions.

$s_\lambda(x)$ = schur function.

$$s_\lambda(x)\, s_\mu(x) = \sum_{\nu \in \Lambda(\infty,\infty)} c_{\lambda\mu}^{\nu}\, s_\nu(x).$$

character stability, degeneration.

Rep Thy. ⟵ - - - - ? - - - - ⟶ Schubert calculus.

<u>Question</u> What can we say about the set

$$NV(d,n) = \{(\lambda,\mu,\nu) \in \Lambda(d,n)^3 : c_{\lambda\mu}^{\nu} > 0\}?$$

<u>II. Theorems</u>.

Saturation: $\lambda \in \Lambda(d,\infty)$ $\quad N\lambda = N\lambda_1 \geq N\lambda_2 \geq \cdots \geq N\lambda_d$.

<u>Theorem</u> (Knutson–Tao '99) $\forall N \in \mathbb{Z}_{>0}$.

$$c_{N\lambda, N\mu}^{N\nu} > 0 \iff c_{\lambda\mu}^{\nu} > 0.$$

Note: $\Leftarrow$ is easy (Borel–Weil: If $L \to X$ has a $GL(d)$-invt section, so does $L^{\otimes N} \to X$)

Horn: $\mathcal{E}(n) = \{(\alpha,\beta,\gamma) : \exists\, A, B, C \in \mathrm{Herm}_{n\times n},\ A+B=C,$

$\mathrm{Eigenvals}(A) = \alpha = \alpha_1 \geq \cdots \geq \alpha_n$

$\mathrm{Eigenvals}(B) = \beta = \beta_1 \geq \cdots \geq \beta_n,\ \mathrm{Eigenvals}(C) = \ldots\ \}$

① **Theorem** (Klyachko '94) $(\alpha,\beta,\gamma)\in\mathcal{E}(n)$

$\iff \sum \alpha_i + \sum \beta_i = \sum \gamma_i$.

and $\forall d<n$ $\forall(\lambda,\mu,\nu)\in NV(d,n)$

$\sum \alpha_{\tilde{\lambda}_i} + \sum \beta_{\tilde{\mu}_i} \geq \sum \gamma_{\tilde{\nu}_i}$ where $\tilde{\lambda}_i = d+1-i+\lambda$

Example $d=1$, $\lambda=\mu=\nu=\phi \leftarrow$ empty partition.

$V_\lambda = V_\mu = V_\nu$ = trivial rep.

$\Longrightarrow \alpha_1 + \beta_1 \geq \gamma_1$.

② **Theorem** $(\lambda,\mu,\nu)\in NV(d,n)$

$\iff |\lambda|+|\mu|=|\nu|$.

and $\forall r<d$ $\forall(\rho,\sigma,\tau)\in NV(r,d)$.

$\sum \lambda_{\tilde{\rho}_i} + \sum \mu_{\tilde{\sigma}_i} \geq \sum \nu_{\tilde{\tau}_i}$

Connections

- Saturation $\Longrightarrow$ (① $\iff$ ②)

  Proof $(\alpha,\beta,\gamma)\in\mathcal{E}(n)$ integral.

  $\overset{\text{symplectic/GIT equivalence}}{\iff}$ $\exists N\in\mathbb{Z}_{>0}$. $(V_{N\alpha}\otimes V_{N\beta}\otimes V^*_{N\gamma})^{GL} \neq 0$

  $\iff c^{N\gamma}_{N\alpha,N\beta} > 0$.

  $\overset{\text{saturation}}{\iff} c^{\gamma}_{\alpha\beta} > 0$.

- ② $\Longrightarrow$ Saturation.

  Proof $c^{N\nu}_{N\lambda\, N\mu} > 0 \iff$ same inequalities hold

  same inequalities!

  $c^{\nu}_{\lambda\mu} > 0 \iff$ same inequalities hold

Prop 2 Let $\varphi : \mathfrak{g}/\mathfrak{p} \xrightarrow{P\text{-equt.}} Z$. If $p_1 T_{\lambda_1}, \dots, p_k T_{\lambda_k}$ transverse. then

$$\sum_{i=1}^{k} \operatorname{codim}(\cancel{p_i} T_{\lambda_i}) \leq \dim Z. \leftarrow \text{explicit, Combinatorial}$$

Problem There ~~may be~~ Sometimes are. no non-trivial φ.

Prop 3 Let $B \subset R \subset P \subset G$.

$$\begin{array}{ccc} P/R & \longrightarrow & G/R \\ p_i \in W_P^R & & \big\downarrow \ \lambda_i p_i \in W_G^R. \\ & & G/P \quad \lambda_i \in W_G^P \end{array}$$

Assume $\prod s_{p_i} \neq 0 \in H^*$

Then

$\prod S_{\lambda_i} \neq 0$

$\Longleftrightarrow \prod S_{\lambda_i p_i} \neq 0.$

Putting Props 1, 2 & 3 together $\Rightarrow$ whenever $\prod s_{p_i} \neq 0$. get necessary inequalities. for $\prod S_{\lambda_i} \neq 0$.

Fact. $G/P = Gr(d,n)$ all Horn inequalities arise in this way.

Misleadingly Simple Example

$G/P = Gr(4,8)$. $k=2$ $\qquad \mathfrak{g}/\mathfrak{p} = Mat_{4\times 4}$

$\Delta(4,8) \ni \lambda = \mu = 3 \geq 3 \geq 2 \geq 0. \longleftrightarrow 23581467 \in W_G^P.$

$p_1 = I_8$ $\qquad p_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$$\begin{pmatrix} * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ * & * & \cdot & \cdot \\ * & * & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * & * \\ & & * & * \\ & & & * \\ & & & * \end{pmatrix} = \begin{pmatrix} * & & & \\ & \ddots & & \\ & & & \\ & & & * \end{pmatrix} \ni v$$

$p_1 T_\lambda$ $\qquad p_2 T_\mu$

Codim $\lambda = 8$

$\therefore\ \sigma_\lambda \sigma_\mu = 0.$

$$
\nu = \begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4. \end{array}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{array}{c} \mu_4 \\ \mu_3 \\ \mu_2 \\ \mu_1 \end{array}
$$

$\Rightarrow$ Horn inequality that gets violated is.

$$
\underset{\underset{3}{\shortparallel}}{\lambda_2} + \underset{\underset{2}{\shortparallel}}{\lambda_3} + \underset{\underset{3}{\shortparallel}}{\mu_2} + \underset{\underset{2.}{\shortparallel}}{\mu_3} \leq 8.
$$

corresponds to $\rho \cancel{= \emptyset} = 1 \geq 1 \in \Lambda(2,4)$ $\quad (\sigma_{\boxminus})^2 \neq 0.$