Introductory Workshop on Combinatorial Representation Theory

Geometric Representation Theory of Affine Hecke Algebras
Syu Kato - 1/23/08

Disclaimer: These notes are meant to augment the slides presented by Syu Kato. (slides attached)

Slide 2 - To clarify the algebra that is being discussed we get that $\forall s \in S$, $s^2 = 1$. The list of things under the stated Coxeter system are parts of the definition (or things that we get for free when we create one). Notice that for the coordinate ring $\mathcal{A}$ the generators are very similar to that of S, however, we impose that $t_s = t_{s'}$ whenever $s \equiv s'$.

Slide 3 - The bottom of this slide explains the connection between their new Hecke algebra and some other well-known ones. For example, we can think of this new Hecke algebra as a t-analog to the Group ring. Notice also that the Iwahori-Hecke algebra is a special case of this new one when we only consider a single parameter t.

Slide 4 - Normally the parameter space $\mathcal{P}$ for an affine Hecke algebra can be very large. However, in the irreducible case, the dimension must be ≤ 3.

Slide 6 - This is a nice realization of $\mathbb{H}_1$ when you consider the geometry, however, it does not say much about the representations.

Slide 8b - Recall that here R is exactly the short roots $(\epsilon_i - \epsilon_j)$ union the long roots $(2\epsilon_i)$.

Slide 9 - Recall that V_2 is not irreducible in this case. It actually splits into the irreducible representation of highest weight $\epsilon_1 + \epsilon_2$ direct sum the trivial representation. So here we're neglecting the trivial representation.

Slide 11 - In the natural $\mathbb{G}$ action on $\mathbb{V}_l$, $(\mathbb{C}^\times)^{l+1}$ acts as scalar multiplication on each factor. Here, $R(T)$ denotes the group ring of T.

Slide 12b - The Hilbert Nullforms are sometimes referred to as the exotic Nilpotent cone. We can tell that we're starting to get somewhere interesting because bipartitions also parametrize the Weyl group.

Slide 14 - Here q_0, q_1, and q_2 are scalar from the field.

Slide 15 - $K^{\mathbb{G}}(\mathcal{Z})$ is the G-equivariant K-theory of the Steinberg Algebra $(\mathcal{Z})$

Slide 20 - Due to time constraints, the lecture essentially ended at this point.

Geometric representation theory of affine Hecke algebras

Syu Kato

skato@msri.org

MSRI/Kyoto U.

Introductory workshop of Combinatorial Representation theory 2008

Parameter spaces of Coxeter groups

(W, S): Coxeter system (S: generator set)

- Coxter diagram (nodes = S, edges = "braid rel.");
- Equiv. relation $\sim$ on S defined by $s \sim s' \Leftrightarrow s \in \mathrm{Ad}(W)s'$;
- Artin braid group $\mathfrak{B}_{(W,S)}$ (generated by $\{T_s\}_{s \in S}$, surjects to W).

The parameter space of (W, S) is:

$$\mathcal{P}_{(W,S)} := \mathrm{Spec}\mathcal{A}_{(W,S)} \cong (\mathbb{C}^{\times})^{\#(S/\sim)},$$

where

$$\mathcal{A} = \mathcal{A}_{(W,S)} := \mathbb{C}[t_s^{\pm 1}; s \in S] / \langle (t_s - t_{s'}); s \sim s' \rangle .$$

Hecke algebra attached to (W, S)

The Hecke algebra $\mathbb{H} = \mathbb{H}_{(W,S)}$ of (W, S) is:

$$\mathbb{H}_{(W,S)} := \mathcal{A}_{(W,S)}[\mathfrak{B}_{(W,S)}]/\mathrm{HR},$$

where HR is the two-sided ideal generated by

(Hecke Relation) $(T_s + 1)(T_s - t_s)$ for $s \in S$.

- $\mathbb{H}/(t_s \equiv 1) \cong \mathbb{C}[W]$ Group ring of W;
- $\mathbb{H}_1 := \mathbb{H}/\langle t_s - t\rangle$ Iwahori-Hecke algebra;
- $\mathbb{H}_{\vec{n}} := \mathbb{H}/\langle t_s - t^{n_s}\rangle$ Lusztig's Hecke algebra with unequal params. (for each $\vec{n} = (n_s)_s \in \mathbb{Z}^S$)

Affine Hecke algebras

Affine Hecke algebras = Hecke algebras of affine Coxeter groups.

Theorem (Classification of irred. affine Coxeter groups)

$\dim \mathcal{P}$		
3	C_n	
2	A_1, B_n	F_4, G_2
1	$A_n\ (n \geq 2), D_n$	E_6, E_7, E_8

Theorem (Affine Hecke algebras of type ABC)

Affine Hecke algebras of type $A_1 BC$ are specialization of type C (up to extention of centers).

Representation theory of affine Hecke algebras of classical type

Previous observation and Clifford theory (cf. Ram-Ramagge) says:

Theorem

Representation theory of affine Hecke algebras of classical types are completely determined by that of $\mathbb{H}_{C_n}$.

Remark

Theorem is not known when restricted to Iwahori-Hecke algebras ($\mathbb{H}_1$). (I expect that the result is in fact negative.)

Geometric Representation theory of affine Hecke algebras

- Kazhdan-Lusztig version of Iwahori-Matsumoto theorem;

1. Realization of $\mathbb{H}_1$ in terms of affine flag variety;
2. Representations are given by the Kazhdan-Lusztig theory of "cells";
3. Equivalence with the next one is the so-called geometric Satake isomorphism.

Geometric Representation theory of affine Hecke algebras

- Kazhdan-Lusztig version of Iwahori-Matsumoto theorem;
- Deligne-Langlands-Lusztig conjecture;

1. Realization of $\mathbb{H}_1$ in terms of the cotangent bundles of finite-dimensional flag variety of dual type;
2. Representations are given by the Springer fibers;
3. A complete classification of irred. $\mathbb{H}_1$-modules to which t does not act by a root of unity;
4. A refined version of the Deligne-Langlands classification of the Iwahori-spherical dual of p-adic Chevalley group by the category equivalence proved by Borel-Matsumoto.

Geometric Representation theory of affine Hecke algebras

- Kazhdan-Lusztig version of Iwahori-Matsumoto theorem;
- Deligne-Langlands-Lusztig conjecture;
- Exotic version of Deligne-Langlands correspondence.

1. A realization $\mathbb{H}_C$ in terms of certain vector bundles of finite-dimensional flag varieties of type C;
2. Representations are given by the “exotic” Springer fibers;
3. A complete classification of irred. $\mathbb{H}_C$-modules to which $\{t_s\}_{s\in S}$ does not act by certain collection of singular values;
4. The name Deligne-Langlands refers to the fact that we do not need Lusztig’s refinement.

Our goals

1. Present formulation/constructions involved in our exotic Deligne-Langlands correspondence;
2. Compare with the (usual) Delgine-Langlands-Lusztig conjecture at each step.

Notation on algebraic groups

$G = Sp(2n, \mathbb{C})$ with its Borel B, and its maximal subtorus T

$$X^*(T) = \bigoplus_{i=1}^{n} \mathbb{Z}\epsilon_i, \text{ char. lattice of } T,$$

where ϵ_i is a basis such that

$$R = \{\pm\epsilon_i \pm \epsilon_j\} \cup \{\pm 2\epsilon_i\} \supset \{\epsilon_i \pm \epsilon_j\}_{i<j} \cup \{2\epsilon_i\} = R^+.$$

$$W_0 := N_G(T)/T \ni s_i \Leftrightarrow \alpha_i = \epsilon_i - \epsilon_{i+1} (i \neq n) \text{ or } 2\epsilon_n$$

Weyl group   simple refl.   simple roots of G

Notation on representations

$\mathfrak{g} = \mathrm{Lie}G$: adjoint representation of G (highest weight $2\epsilon_1$)

$V_1 := \mathbb{C}^{2n}$: vector representation of G (highest weight ϵ_1)

$V_2 := \wedge^2 V_1$: alternating representation of G (highest weight $\epsilon_1 + \epsilon_2$)

$\mathbb{V}_\ell := V_1^{\oplus \ell} \oplus V_2$: ℓ-exotic representation ($\ell = 1, 2$)

Lemma (Weight distribution of $\mathbb{V}_\ell$)

We have the following correspondence:

$\mathbb{V}_\ell$				$\mathfrak{g}$
V_1	$\pm\epsilon_i$	$\leftrightarrow$	$\pm 2\epsilon_i$	**long roots**
V_2	$\pm\epsilon_i \pm \epsilon_j$	$\leftrightarrow$	$\pm\epsilon_i \pm \epsilon_j$	**short roots**

Here we neglect weight zero part.

The main point of the whole story

Regard $\mathbb{V}_2$ as an analogue of $\mathfrak{g}$

(in the Deligne-Langlands-Lusztig conjecture)

Remark

1. $\mathbb{V}_\ell$ admits an action of $(\ell + 1)$-dimensional torus;
2. $X^*(T)$ is isomorphic to the coroot lattice of G;
3. Geometry is not symplectic, but Calabi-Yau.

Additional notation

$\mathbb{G} := G \times (\mathbb{C}^\times)^{\ell+1}$

- $\mathfrak{g}$ admits natural $\mathbb{G}$-action for $\ell = 0$;
- $\mathbb{V}_\ell$ admits natural $\mathbb{G}$-action.

$\mathfrak{g}^+ = \mathrm{Lie}[B, B]$: the nilradical of B;
$\mathbb{V}^+ = \mathbb{V}_\ell^+ :=$ sum of T-eigensps. of $\mathbb{V}_\ell$ of weights $\mathbb{Q}_{>0}R^+$.
($\mathbb{V}^+$ does not contain 0-weight space)

$\mathcal{T} := G \times^B \mathfrak{g}^+$: cotangent bundle of G/B;
$F = F_\ell := G \times^B \mathbb{V}_\ell^+$: its exotic analogue.

$$R(T) \ni e^\lambda \Leftrightarrow \mathcal{L}_\lambda \in \mathrm{Pic} G/B \cong \mathrm{Pic}\mathcal{T} \cong \mathrm{Pic} F.$$

The scheme of Hilbert nullforms

$$\mathcal{N} := \{\xi \in \mathfrak{g}; P(v) = 0 \text{ for all } P \in \mathbb{C}[\mathfrak{g}]^G_+\} = \{\xi \in \mathfrak{g}; \mathrm{ad}_{\mathfrak{g}}\xi \text{ is nilpotent endo.}\}$$

the nilpotent cone of $\mathfrak{g}$.

Theorem (Basic properties)

- The variety $\mathcal{N}$ is normal with $\mathbb{G}$-action for $\ell = 0$ (Kostant);
- The set of G-orbits of $\mathcal{N}$ is parameterized by symbols (Lusztig);

The scheme of Hilbert nullforms

$$\mathcal{N} := \{\xi \in \mathfrak{g}; P(v) = 0 \text{ for all } P \in \mathbb{C}[\mathfrak{g}]_+^G\} = \{\xi \in \mathfrak{g}; \mathrm{ad}_{\mathfrak{g}}\xi \text{ is nilpotent endo.}\}$$

the nilpotent cone of $\mathfrak{g}$.

$$\mathfrak{N} = \mathfrak{N}_\ell := \{v \in \mathbb{V}_\ell; P(v) = 0 \text{ for all } P \in \mathbb{C}[\mathbb{V}_\ell]_+^G\}$$

the scheme of Hilbert nullforms of $\mathbb{V}_\ell$.

Theorem (Basic properties)

- The variety $\mathcal{N}$ is normal with $\mathbb{G}$-action for $\ell = 0$ (Kostant);
- The set of G-orbits of $\mathcal{N}$ is parameterized by symbols (Lusztig);
- The variety $\mathfrak{N}_\ell$ is normal with $\mathbb{G}$-action (Schwarz);
- The set of G-orbits of $\mathfrak{N}_1$ is parameterized by bi-partitions of n.

Steinberg-type varieties

Theorem (Springer-Hesselink)

- The moment map $\mu : \mathcal{T} \longrightarrow \mathcal{N}$ is a resolution of singularity;

We define

$$\mathcal{Z} := \mathcal{T} \times_{\mathcal{N}} \mathcal{T}$$

(the Steinberg variety)

The cartesian product have two projections $p_i : \mathcal{Z} \longrightarrow \mathcal{T}$ $(i = 1, 2)$

Steinberg-type varieties

Theorem (Springer-Hesselink)

- The moment map $\mu : \mathcal{T} \longrightarrow \mathcal{N}$ is a resolution of singularity;
- The natural map $\nu : F \ni (gB, X) \mapsto X \in \mathfrak{N}$ gives a resolution of singularity of $\mathfrak{N}$.

We define

$$Z = Z_\ell := F_\ell \times_{\mathfrak{N}_\ell} F_\ell, \qquad (\mathcal{Z} = \mathcal{T} \times_{\mathcal{N}} \mathcal{T})$$

(analogue of Steinberg variety and original)

$$p_i : Z \ni (g_1 B, g_2 B, X) \to (g_i B, X) \in F \ (i = 1, 2)$$

Reminder of the group action

For $\ell = 2$, the varieties and maps

$$Z \xrightarrow{p_1, p_2} F \xrightarrow{\nu} \mathfrak{N}$$

admits a natural $\mathbb{G}$-action coming from

$$\begin{aligned}\mathbb{G} \times F \ni (s, q_0, q_1, q_2) \times (gB, \overbrace{X_0 \oplus X_1 \oplus X_2}^{X}) \\ \mapsto (sgB, q_0^{-1} sX_0 \oplus q_1^{-1} sX_1 \oplus q_2^{-1} sX_2) \in F.\end{aligned}$$

The cases $\ell = 1$ and $\mathcal{Z} \to \mathcal{T} \to \mathcal{N}$ are similar.

Convolution construction of algebras

Theorem (Ginzburg)

The assignment

$$\star : K^{\mathbb{G}}(\mathcal{Z}) \otimes K^{\mathbb{G}}(\mathcal{T}) \ni ([\mathcal{K}], [\mathcal{E}]) \mapsto \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_1)_* (\mathcal{K} \otimes p_2^* \mathcal{E})] \in K^{\mathbb{G}}(\mathcal{T})$$

equip a unital ring structure of $K^{\mathbb{G}}(\mathcal{Z})$ with a faithful representation $K^{\mathbb{G}}(\mathcal{T})$.

Remark

$K^{\mathbb{G}}(\mathcal{Z})$ contains $R(\mathbb{G})$ as its center. (tensor product)

Convolution construction of algebras

Theorem (Ginzburg)

The assignment

$$\star : K^{\mathbb{G}}(Z) \otimes K^{\mathbb{G}}(F) \ni ([\mathcal{K}], [\mathcal{E}]) \mapsto \sum_{i \geq 0} (-1)^i [\mathbb{R}^i (p_1)_* (\mathcal{K} \otimes p_2^* \mathcal{E})] \in K^{\mathbb{G}}(F)$$

equip a unital ring structure of $K^{\mathbb{G}}(Z)$ with a faithful representation $K^{\mathbb{G}}(F)$.

Remark

$K^{\mathbb{G}}(Z)$ contains $R(\mathbb{G})$ as its center. (tensor product)

The Lusztig realization of $\mathbb{H}_1$

Theorem (Lusztig 1984)

We have an isomorphism

$$\mathbb{H}_1 \xrightarrow{\cong} \mathbb{C} \otimes_{\mathbb{Z}} K^{\mathbb{G}}(\mathcal{Z}),$$

where $t = t_s$ $(s \in S)$ is identified with the degree-one character $\mathbf{q}$ of

$$\mathbb{C}^\times \hookrightarrow G \times (\mathbb{C}^\times)^{0+1} = \mathbb{G}.$$

Remark

1. Lusztig's construction works for ALL (irreducible) affine Hecke algebras by replacing G by adjoint semi-simple algebraic groups;
2. If we forget $\mathbb{C}^\times$-action, we obtain Lusztig version of Springer realization of $\mathbb{C}[W_0]$.

Exotic realization of affine Hecke algebras

Theorem (math.RT/0601155 (now revising), theorem A)

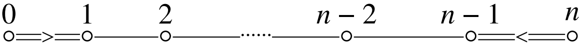
For $\ell = 2$, we have an isomorphism

$$\mathbb{H}_{C_n} \xrightarrow{\cong} \mathcal{A} \otimes_{R((\mathbb{C}^\times)^{\ell+1})} K^{\mathbb{G}}(Z),$$

where the inclusion $R((\mathbb{C}^\times)^{\ell+1}) \hookrightarrow \mathcal{A}$ is given as:

$$\mathbf{q}_2 = t_1 = t_2 = \cdots = t_{n-1}$$
$$-\mathbf{q}_0\mathbf{q}_1 = t_n, \quad -\mathbf{q}_0/\mathbf{q}_1 = t_0,$$

where $\mathbf{q}_i$ are deg 1 chars of $V_1 \oplus V_1 \oplus V_2$ and



Comments on the proof

The proof is basically common between the case $\mathcal{Z}$ (original case) and the case Z (exotic case). (The point was to employ $\mathbb{V}_2$.)

- $K^{\mathbb{G}}(Z) = \left\langle T_i^g, e^\lambda; 1 \le i \le n \right\rangle$;
- Compute the action of T_i^g on $K^{\mathbb{G}}(F)$. Same as the "usual" one except for $i = n$;
- We have
$$T_n^g e^\lambda = (1 - \mathbf{q}_0 e^{\epsilon_n})(1 - \mathbf{q}_1 e^{\epsilon_n}) \frac{e^\lambda - e^{s_n\lambda - 2\epsilon_n}}{1 - e^{-2\epsilon_n}};$$
- Identify $K^{\mathbb{G}}(F)$ with the anti-spherical representation of $\mathbb{H}$. (cf. Macdonald's recent book)

Rough idea of construction of representations

If $\mathcal{E}$ is a $\mathbb{G}$-equivariant coherent sheaf on $\nu^{-1}(\overline{\mathcal{O}})$ for a $\mathbb{G}$-orbit $\mathcal{O} \subset \mathfrak{N}$, then

$$p_1(p_2^{-1}(\overline{\mathcal{O}})) \subset \nu^{-1}(\overline{\mathcal{O}})$$

implies that $K^{\mathbb{G}}(\nu^{-1}(\overline{\mathcal{O}}))$ admits $K^{\mathbb{G}}(Z)$-module structure.

⇒ one hope to obtain enough $K^{\mathbb{G}}(Z)$-modules to classify irreducible representations of $K^{\mathbb{G}}(Z)$.

Rough idea of construction of representations

If $\mathcal{E}$ is a $\mathbb{G}$-equivariant coherent sheaf on $\nu^{-1}(\overline{\mathcal{O}})$ for a $\mathbb{G}$-orbit $\mathcal{O} \subset \mathfrak{N}$, then

$$p_1(p_2^{-1}(\overline{\mathcal{O}})) \subset \nu^{-1}(\overline{\mathcal{O}})$$

implies that $K^{\mathbb{G}}(\nu^{-1}(\overline{\mathcal{O}}))$ admits $K^{\mathbb{G}}(Z)$-module structure.

$\Rightarrow$ one hope to obtain enough $K^{\mathbb{G}}(Z)$-modules to classify irreducible representations of $K^{\mathbb{G}}(Z)$.

This is too naive to obtain a correct answer. It is a miracle that this naive expectation works after taking a fixed point set by a suitable semi-simple elements.

Bernstein center

Theorem (Bernstein-Lusztig)

The center of the algebra $\mathbb{H}$ is isomorphic to

$$\mathcal{A} \otimes_{\mathbb{Z}} R(G) \cong \mathcal{A}(T)^{W_0}.$$

Remark

1) $\mathbb{H}$ is a free $Z(\mathbb{H})$-module of rank $(\#W_0)^2$;
2) This follows from "Bernstein presentation" of $\mathbb{H}$.

Preparation for the statement

Put

$$\vec{q} := \begin{cases} (q_0, q_1, q_2) & (\ell = 2, \text{ exotic case}) \\ q & (\ell = 0, \text{ usual DLL case}) \end{cases}$$

Semi-simple element $a := (s, \vec{q}) \in \mathbb{G}$ defines a character

$$a : R(\mathbb{G}) \ni [V] \mapsto \mathrm{tr}(a, V) \in \mathbb{C} =: \mathbb{C}_a$$

We define

$$\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H}$$

and

$$\mathbb{G}(a) := Z_{\mathbb{G}}(a) \circlearrowright \mathfrak{N}^a = \mathfrak{N}_2^a.$$

Similar consideration yields $\mathbb{H}_{1,a}$ and $\mathbb{G}(a) \circlearrowright \mathcal{N}^a$ (for DLL conj).

Exotic Deligne-Langlands correspondence

Theorem (math.RT/0601155, revised version of Theorem E)

Let $a := (s, q_0, q_1, q_2) \in \mathbb{G}$ be a semi-simple element s.t.:

- q_2 is not a root of unity of order $\leq 2n$;

Exotic Deligne-Langlands correspondence

Theorem (math.RT/0601155, revised version of Theorem E)

Let $a := (s, q_0, q_1, q_2) \in \mathbb{G}$ be a semi-simple element s.t.:

- q_2 is not a root of unity of order $\le 2n$;
- $q_0 q_1^{\pm} \neq q_2^m$ for some integer m s.t. $|m| \le n$.

Exotic Deligne-Langlands correspondence

Theorem (math.RT/0601155, revised version of Theorem E)

Let $a := (s, q_0, q_1, q_2) \in \mathbb{G}$ be a semi-simple element s.t.:

- q_2 is not a root of unity of order $\leq 2n$;
- $q_0 q_1^{\pm} \neq q_2^m$ for some integer m s.t. $|m| \leq n$.

Then we have

$$\mathrm{Irrep}\mathbb{H}_a \overset{1:1}{\leftrightarrow} \mathbb{G}(a) \backslash \mathfrak{N}_2^a.$$

Exotic Deligne-Langlands correspondence

Theorem (math.RT/0601155, revised version of Theorem E)

Let $a := (s, q_0, q_1, q_2) \in \mathbb{G}$ be a semi-simple element s.t.:

- q_2 is not a root of unity of order $\le 2n$;
- $q_0 q_1^{\pm} \neq q_2^m$ for some integer m s.t. $|m| \le n$.

Then we have

$$\mathrm{Irrep}\mathbb{H}_a \overset{1:1}{\leftrightarrow} \mathbb{G}(a)\backslash\mathfrak{N}_2^a.$$

Remark

When the conditions holds, $\mathfrak{N}_2^a$ is an affine space with finitely many $\mathbb{G}(a)$-orbits.

Comments on the statement

- We have $\mathbb{H}_{C_n}/(\mathbf{q}_0 + \mathbf{q}_1) \cong \widehat{\mathbb{H}}_{B_n}$. This gives

$$-1, \text{ or } -q_0^2 = q_0 q_1^{\pm} \neq q_2^m$$

  as the coverage of the theorem.
  I.e. our theorem covers $\mathbb{H}_{\vec{n}}$ of type B or C (if $|t| \neq 1$).

- Putting $t_i = t$ does not imply Kazhdan-Lusztig theorem (the DLL conjecture);

- For $n = 2$, our result recovers Enomoto's classification (J Math Kyoto U 2006)

- In general, the RHS consist of ∞-orbits...

- With an aid of Ginzburg-Lusztig theory (cf. Chriss-Ginzburg), the proof is given by developing the geometry of $\mathfrak{N}$.

Example $(n = 1)$

We have $\mathbb{H}_{C_1} = \mathbb{H}_{A_1}$

$V_2 \cong \mathbb{C}$ does not contribute to F or Z

$\Rightarrow q_2$ does not define an essential parameter of $\mathbb{H}$.

For each $a \in \mathbb{G}$, $|\mathrm{Irr}\mathbb{H}_a| \leq 2$

# of orbits	1	2	3	4
# of reprs.	1	2	2	2
Condition		$q_0 \neq q_1^{\pm 1}$	$q_0 = q_1^{-1}$	$q_0 = q_1$

This table shows we really need the condition like "$q_0 q_1^{\pm 1} \neq q_2^m$".

Comparison of correspondences

Let $a \in \mathbb{G}$ be a semi-simple element. We have

$\mathbb{C}[W] \cong \mathbb{H}_{1,a}$, or $\mathbb{H}_a$ if

$$a = (s, \vec{q}) = \begin{cases} (1,1) & \text{(if we consider } \mathbb{H}_1 \text{ : DLL case)} \\ (1,1,-1,1) & \text{(if we consider } \mathbb{H} \text{ : eDL case)} \end{cases}$$

The inclusion $\mathbb{C}[W_0] \subset \mathbb{C}[W]$ gives:

Theorem (Springer-Kazhdan-Lusztig-Ginzburg)

1. $B_x := H_*(\mu^{-1}(x))$ is a graded $\mathbb{C}[W_0]$-module for $x \in \mathfrak{N}_2^a$.
2. $E_y := H_*(\nu_1^{-1}(y))$ is a graded $\mathbb{C}[W_0]$-module for $y \in \mathfrak{N}_1 \cong \mathfrak{N}_2^a$.

Springer correspondences

Statement

Theorem (Springer 1976)

The assignment

$$G\backslash\mathcal{N} \ni x \mapsto B_x^{top} \in W_0 - \text{mod}$$

sets up to a surjection

$$\text{Irrep}W_0 \longrightarrow\!\!\!\!\!\rightarrow G\backslash\mathcal{N}.$$

Here B_x^{top} denote the top degree part of B_x coming from the degree of homology groups.

Springer correspondences

Statement

Theorem (Springer 1976)

The assignment

$$G\backslash\mathcal{N} \ni x \mapsto B_x^{top} \in W_0 - \text{mod}$$

sets up to a surjection

$$\text{Irrep} W_0 \longrightarrow\!\!\!\!\!\rightarrow G\backslash\mathcal{N}.$$

Here B_x^{top} denote the top degree part of B_x coming from the degree of homology groups.

Remark

This surjection can be lifted to a bijection if one take account into A-group data (some representations of fundamental groups of orbits of $\mathcal{N}$).

Springer correspondences

Statement

Theorem (math.RT/0607478, Thm C)

The assignment

$$G\backslash\mathfrak{N}_1 \ni y \mapsto E_y^{top} \in W_0 - \mathrm{mod}$$

sets up a bijection

$$\mathrm{Irrep} W_0 \overset{1:1}{\leftrightarrow} G\backslash\mathfrak{N}_1.$$

Here E_y^{top} denote the top degree part of E_y coming from the degree of homology groups.

Springer correspondences

Statement

Theorem (math.RT/0607478, Thm C)

The assignment

$$G\backslash\mathfrak{N}_1 \ni y \mapsto E_y^{top} \in W_0 - \text{mod}$$

sets up a bijection

$$\text{Irrep} W_0 \overset{1:1}{\leftrightarrow} G\backslash\mathfrak{N}_1.$$

Here E_y^{top} denote the top degree part of E_y coming from the degree of homology groups.

Remark

This bijection represents the fact that fundamental groups of orbits of $\mathfrak{N}_1$ are all trivial.

Springer correspondences

Example ($n = 2$)

We summarize usual and exotic Springer correspondences as:

$W_0^\vee$	dim	$G\backslash\mathfrak{N}$	$G\backslash\mathcal{N}$	$A(X)$	Char.
sign	1	$\{0\}$	$\{0\}$	$\{1\}$	
Ssign	1	$\mathbf{y}[2\epsilon_1]$	$\mathbf{x}[2\epsilon_1]$	$\{1\}$	
Lsign	1	$\mathbf{y}[\alpha_1]$	$\mathbf{x}[\alpha_1]$	$\{\pm 1\}$	-1
regular	2	$\mathbf{y}[\alpha_1] + \mathbf{y}[\epsilon_1]$	$\mathbf{x}[\alpha_1]$	$\{\pm 1\}$	1
triv	1	$\mathbf{y}[\alpha_1] + \mathbf{y}[\epsilon_2]$	$\mathbf{x}[\alpha_1] + \mathbf{x}[2\epsilon_2]$	$\{1\}$	

where $\mathbf{y}[\beta] \in \mathbb{V}_1$, and $\mathbf{x}[\beta] \in \mathfrak{g}$ are weight β-bases.

The Deligne-Langlands-Lusztig conjecture

Theorem (Kazhdan-Lusztig 1987, Xi 2007)

Let $a := (s, q) \in \mathbb{G}$ be a semi-simple elmenent s.t.:

- q is not a root of the Poincare polynomial of (W, S).

Then we have

$$\mathrm{Irrep}\mathbb{H}_a \twoheadrightarrow \mathbb{G}(a)\backslash\mathcal{N}^a.$$

This can be lifted to a one-to-one correspondence if we take account into the A-group data (= the Lusztig part of the DLL parameter).

Remark

For type A, we completely know what happens even if q is a small root of unity. (Lusztig, Ariki, Grojnowski, Vazirani,...) Back to comments