1/24/08 Andrei Okounkov # 3

recall we're discussing Λ^* = alg. of poly's on partitions
it has 3 different bases
S^*_μ – interpolating schur functions, $\{f_\mu\}$ – central characters
$\{p^*_\mu\}$ centers...

by Fourier trans $\Lambda^* \cong \bigoplus_{k\geq 0} \mathbb{Z}\mathbb{Q}S(k)$
recall $p^*_k(\lambda) = \sum_i [(\lambda_i - i + \frac{1}{2})^k - (-i+\frac{1}{2})^k]$

today we want to understand how to write f_μ's in terms of p^*_μ
we'll try to create a ficticious element in $\underline{\bigoplus \ldots}$
s.t. $p^*_k(\lambda)$ is the central character
of the completed k-cycle $\frac{1}{k}C_{(k)}$ + smaller permutations

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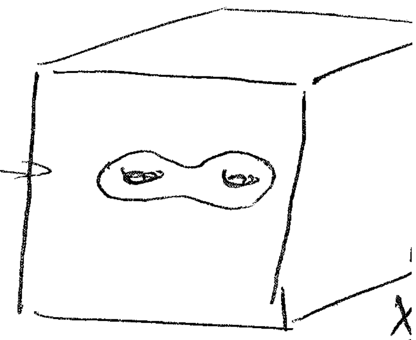
|Gromov – Witten – ... – ... – ... – ... theory|

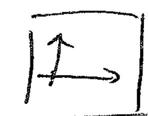
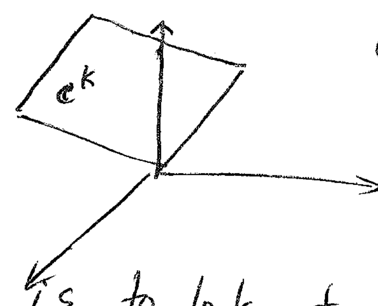
Start w/ var. X smooth proj.
we want to understand alg. curves inside
$\overline{M}_g(X,d)$

C  $\xrightarrow{f}$ 

curve genus g map to X

$f_*([C]) = d \in H_2(X)$
a two-cycle

Grassmannians instead think of vector space ~~Gr(k,n)~~
Gr(k,n)

$\mathbb{C}^k$  $\xrightarrow{L}$  $\mathbb{C}^n$

So the Goal of G-W-... theory is to look at
Schubert calculus in the setting above...
i.e. define cohomology class and compute intersections.

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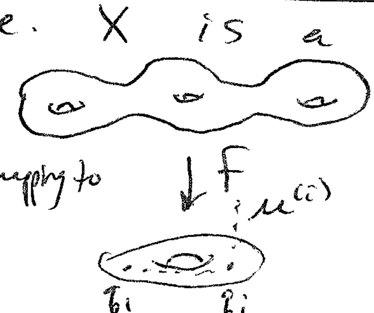
for today dim X = 1 i.e. X is a curve
given smooth curve



} a branched covering

then the topology uniquely defines an algebra

(unique structure upstairs makes this map holomorphic

locally over a pt. 
1 ← some unramified
3 ← some ramified.

to every pt. $q \in X$ we associate a partition of d. ← called profile or monodromy
(each component in preimage is a part)

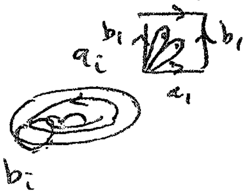
Hurwitz problem: enumerate all branched covers with given monodromy of cycle type $\mu^{(1)}, \dots, \mu^{(n)}$

$$\text{Hurwitz}(\mu^{(1)}, \dots, \mu^{(n)}; \text{genus of } X) \overset{\text{by def}}{=}$$

$$\sum_{\substack{\text{branched covers } f \\ \text{w/ this monodromy}}} \frac{1}{|\mathrm{Aut}\, f|} = \frac{\#\text{ homomorphisms } \pi_1(X \setminus \{q_i\}) \longrightarrow S(d) \text{ s.t. } \circlearrowleft_{q_i} \longmapsto \text{conj. class } C_{\mu^{(i)}}}{|S(d)|}$$

(Branched cover means an unramified cover when the branch pts. are removed.)

h = genus of X



$$= \frac{1}{d!}\left(\#\text{ of solutions to } \sigma_1 \sigma_2 \cdots \sigma_n a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} = 1 \text{ s.t. } \sigma_i \in C_{\mu^{(i)}}\right)$$

reg. rep

So if $\underline{h=0}$ then we get $= \dfrac{1}{(d!)^2} \operatorname{tr}_{\mathbb{Q}S(d)} \prod C_{\mu^{(i)}}$

for $h>0$ $\cdots(\ )^{2-2h}\cdots$
$$= \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^2 \cdot \prod f_{\mu_i}(\lambda)$$ ← a central character

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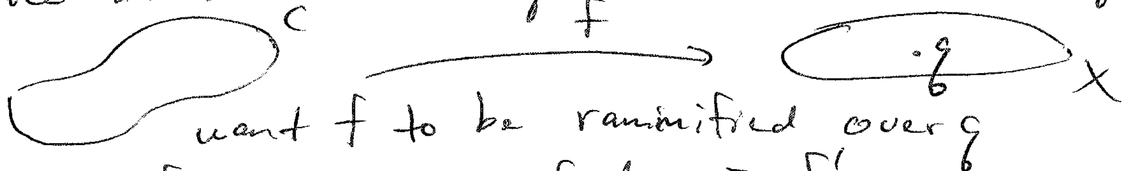
~~for $h>0$~~ So how do we make Alg. Geom. out of Hurwitz prob??
(now we need to look at equations ...)
Var. Y, write down equation
think about $[x:y:z]$ - coord (projective) and $x^3+y^3+z^3=0$.
its a section of $\mathcal{O}(3)$. (eqn's are sections of linebundles)

How do we write an equation for "ramified over q"?



$$C \xrightarrow{f} X$$

want f to be ramified over q

clearly, $f(p) = q$ ramified $\Rightarrow f'(p) = 0$.

So $f'(p) \in T_p^* C$ ← cotangent line to C at p.

(i.e. $\forall p$, $T_p^* C$ is a line ...?)

$T^*_p C$ is a line bundle over $\overline{M}_{g,1}(X,d)$ call it $\mathcal{L}$

(the 1 stands for one marked pt. e.g. p)

if we pick a section the locus where it vanishes minus locus where it blows up is called a Chern-class

So $\{f'(p) = 0\} = c_1(\mathcal{L})$ ← first chern class of $\mathcal{L}$

on the smooth locus...

(an elliptic curve such as $x^3+y^3+z^3=0$ is a rep. of the 1^{st} Chern class)

Now, how do we say f has triple ramification?

$f'(p) = 0$ and $f''(p) = 0$. (second deriv)

section of $\mathcal{L}$ — section of $\mathcal{L} \otimes \mathcal{L}$

So let's take $c_1(\mathcal{L}) \cap c_1(\mathcal{L}^2)$ ✓ but this is $2c_1(\mathcal{L})$

So $= 2c_1^2$.

How bout 5-tuple rammification? $4!\, c_1^4$

The (k)-cycle ramm. over q_i condition of the smooth part of $\overline{M}_{g,n}(X,d)$ is a representative of the $(k-1)!\, c_1(\mathcal{L}_i)^{k-1}$. $\mathcal{L}_i = T_{p_i}^* C$

On the whole of $\overline{M}_{g,n}(X,d)$

$$c_1(\mathcal{L})^k = \frac{1}{(k+1)!} \cdot \text{completed } (k+1)\text{-cycle}.$$

$\boxed{\text{Theorem}}$ (Okounkov – Pandharipande)

$$\int_{[\overline{M}_{g,n}(X,d)]^{\text{virt.}}} \prod c_1(\mathcal{L}_i)^{k_i} = \sum_{|\lambda|=d} \left(\frac{\dim \lambda}{d!}\right)^{2-2\,\text{genus}(X)} \prod \frac{P^*_{k_i+1}(\lambda)}{(k_i+1)!}$$

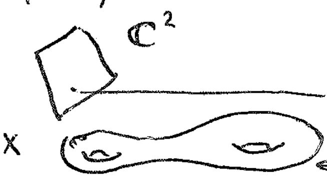
virtually...?

From the modern perspective this is one of the simplest instances of the GW/DT correspondence.

(formulas like the one above appear in the geometry of the Hilbert scheme of pts. of $\mathbb{C}^2$.)

the correspondence has to do w/ the case when $\dim X = 3$.

i.e. consider $X =$



study curves that project to curves here and pts. in $\mathbb{C}^2$.