Introductory Workshop on Combinatorial Representation Theory

Total Positivity for Flag Varieties: combinatorics, topology, and toric geometry
Lauren Williams - 1/24/08

Disclaimer: These notes are meant to augment the slides presented by Lauren Williams. (slides attached)

Slide 5 - The question in bullet 2 has been answered in the case when we assume that one cell is contained in the other.

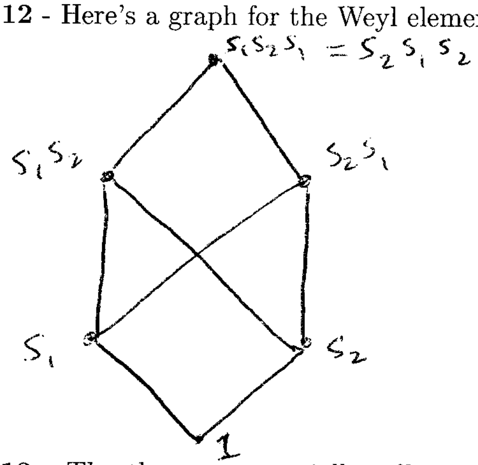
Slide 6 - We can think of an element as a $2 \times n$ matrix. $\begin{bmatrix} \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \end{bmatrix}$ where the columns are labeled by $v_1, ..., v_n$ the basis for V.

Slide 7 - We can make all of the same descriptions as we have here for G/P (the partial flag variety). Define $x_i(t) = I + te_{i,i+1}$ and $y_i(t) = I + te_{i+1,i}$ for any number t.

Slide 8 - Bullet 3 describes how to decompose. Bullet 4 explains how we can get a decomposition for the flag variety by intersecting.

Slide 10 - The +'s in these diagrams represent copies of $\mathbb{R}^+$, so if you count the number of +'s in each diagram, this tells you the dimension of the corresponding cell. The numbers underneath each diagram are shorthand for Weyl group elements. (i.e. 13 is shorthand for $s_1 s_3$.)

Slide 12 - Here's a graph for the Weyl elements when the Weyl group is $W = S_3$.



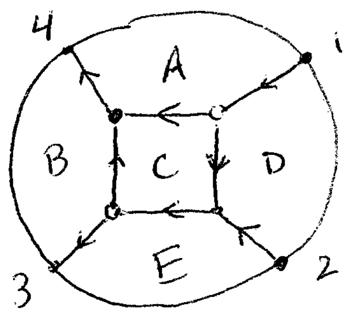
Slide 13 - The theorem essentially tells us that we can always synthetically construct a CW-complex with the chosen face poset.

Slide 15 - a plebic graph is a bicolored graph that is embedded in a disc. Here's an example. (A good example has been included in the note for slide 20 below)

Slide 16 - For example, we denote the first permutation matrix $\begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$ by s_1

Slide 19 - There is a nice way to start with a parameter and extend to an attaching map.

Slide 20 - Here's an example explaining how to start with a plebic graph and use it to determine a map onto an element in the positive part of the Grassmannian $(Gr_{2,4})_{>0}$.



This is a directed graph, hence paths must follow these directions.

In this case, $(A, B, C, D, E) \mapsto \begin{bmatrix} 1 & 0 & -(DE) & -(BED + BCDE) \\ 0 & 1 & E & BE \end{bmatrix}$ each of the entries comes from the product of all of the regions to the left of a path from one vertex to another. e.g. the 2, 3-entry is a E as there is only one path from vertex 2 to vertex 3 and it only the region E is on the left. (the 1, 4 entry has a sum of two terms because there are two different paths from 1 to 4) Finally, the signs on each entry are chosen to guarantee that each of the minors has a positive determinant.

Slide 21 - Recall that $\rho = \frac{1}{2} \sum$ positive roots

Slide 24 - This is useful for computations because if a Morse matching has only one critical cell of dimension zero, then the space is contractible.

Total positivity for flag varieties: combinatorics, topology, toric geometry

Lauren K. Williams, MSRI and Harvard

MSRI Introductory workshop on Combinatorial Representation Theory, January 2008

Program

0. Introduction and motivation for total positivity

1. Totally nonnegative flag varieties $(G/P)_{\geq 0}$

2. The poset of cells in Rietsch's cell decomposition of $(G/P)_{\geq 0}$

3. A combinatorial result and a conjecture

4. A related family of toric varieties ... $(G/P)_{\geq 0}$ is a CW complex

5. Using discrete Morse theory to prove topological results

Total Positivity

A square matrix is totally positive (resp. nonnegative) if all of its minors are positive (resp. nonnegative).

Since early 1900's: much study of these matrices, by Schoenberg, Whitney, Gantmacher-Krein, Karlin, etc: results on spectral properties and parameterizations; connections to oscillating mechanical systems and Markov processes.

Late 1990's: Lusztig generalized the theory of total positivity. He introduced the totally positive part of a reductive algebraic group: for GL_n, the resulting $GL_n(\mathbb{R}_{>0})$ recovers the classical notion of totally positive matrices.

Additionally, he introduced the totally positive part $(G/P)_{>0}$ of a flag variety G/P, and its closure, the totally nonnegative part $(G/P)_{\geq 0}$ – a "remarkable polyhedral subspace."

Total Positivity

Total positivity has connections to:

1. canonical bases
2. cluster algebras
3. tropical geometry

Additionally, the combinatorics involved points to connections with:

1. the asymmetric exclusion process
2. quantum algebras
3. lots of other things!

Totally nonnegative flag varieties $(G/P)_{\geq 0}$

• Lusztig's constructions came from his theory of canonical bases. Using this machinery, he proved that $(G/P)_{\geq 0}$ is contractible.

• Konni Rietsch proved that $(G/P)_{\geq 0}$ has a natural structure of a cell complex (conjectured by Lusztig), and described the poset of (closures of) cells.

• However, at that time, many questions about $(G/P)_{\geq 0}$ were unanswered. Is the cell decomposition a CW complex? What about the topology of the closures of individual cells? E.g. are they contractible?

• GOAL: to apply combinatorial techniques to the poset of cells to try to understand what $(G/P)_{\geq 0}$ looks like. Then use toric geometry and more combinatorics to answer the above questions.

Totally nonnegative flag varieties $(G/P)_{\geq 0}$

• The Grassmannian is:

$$Gr_{kn}(\mathbb{R}) = \{V \subset \mathbb{R}^n \,|\, \dim V = k\}$$
$$= \mathrm{GL}_k \setminus \mathrm{Mat}(k,n)$$

• The *totally nonnegative part* is:

$(Gr_{kn})_{\geq 0}$ = subset of Gr_{kn} where max. minors (Plucker coord's) ≥ 0

• Description for $(Gr_{2n})_{\geq 0} \ldots$

• Cell decomposition: cells are determined by specifying which minors are strictly positive and which are 0.

• Similar description for other flag varieties in terms of minors.

The Lie-theoretic description of $(G/B)_{\geq 0}$

- Let G be a semisimple linear algebraic group over $\mathbb{C}$ split over $\mathbb{R}$.
- Fix a *pinning* $(T, B^+, B^-, x_i, y_i; i \in I)$ for G.
- For SL_n, the standard pinning is: diagonal matrices, upper-triangular and lower-triangular matrices, and simple root subgroups $x_i(t) = I_n + tE_{i,i+1}$ and $y_i(t) = I_n + tE_{i+1,i}$.
- Let U^+ and U^- be the unipotent radicals of B^+ and B^- (upper and lower unitriangular matrices).
- The totally nonnegative part $U^-_{\geq 0}$ of U^- is the subgroup in U^- generated by the $y_i(t)$ for $t \in \mathbb{R}_{\geq 0}$.
- Identify the flag variety G/B^+ with the variety of Borel subgroups $\{g \cdot B^+ \mid g \in G\}$ where $g \cdot B^+ := gB^+g^{-1}$.
- The totally nonnegative part of G/B^+ is defined to be $(G/B^+)_{\geq 0} := \overline{\{u \cdot B^+ \mid u \in U^-_{\geq 0}\}}$.

Cells of $(G/B)_{\geq 0}$

• The *Weyl group* is $W := N_G(T)/T$. For SL_n, W is the symmetric group.

• We have the *Bruhat decompositions*

$$G/B^+ = \sqcup_{w \in W} (B^+ w) \cdot B^+ = \sqcup_{w \in W} (B^- w) \cdot B^+$$

into B^+-orbits (*Bruhat cells*) and B^- orbits (*opposite Bruhat cells*).

• For $v, w \in W$ define $R_{v,w} := (B^+ w) \cdot B^+ \cap (B^- v) \cdot B^+$. Non-empty iff $v \leq w$ (Bruhat order), and then irreducible of dimension $\ell(w) - \ell(v)$. *R-polynomials of Kazhdan-Lusztig* ...

• For $v, w \in W$ with $v \leq w$, let $R_{v,w;>0} := R_{v,w} \cap (G/B)_{\geq 0}$. This is a cell (i.e. homeomorphic to an open ball); these cells give a cell decomposition of $(G/B)_{\geq 0}$ (Rietsch).

• There are natural generalizations to partial flag varieties ...

The poset of cells of $(G/B)_{\geq 0}$

Question: when is a cell $\mathcal{R}_{x,w;>0}$ contained in the closure of another cell?

Answer (Rietsch): $\mathcal{R}_{x,w;>0} \subset \overline{\mathcal{R}_{x',w';>0}}$ if and only if $x' \leq x \leq w \leq w'$ (Bruhat order).

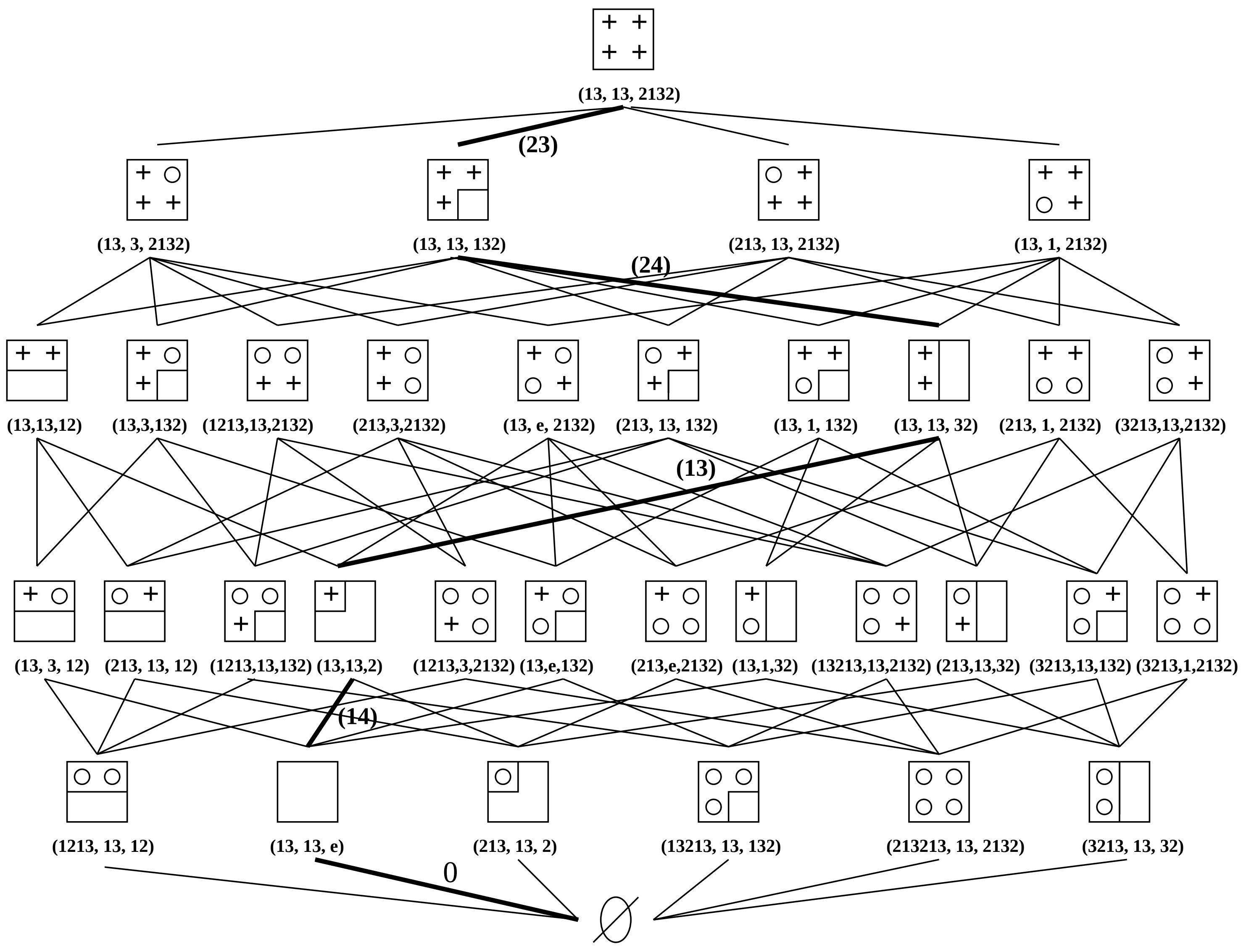
We refer to this abstract poset as Q. It is the *face poset* of $(G/B)_{\geq 0}$.

There is a generalization to partial flag varieties G/P_J ... here $J \subset I$ and I is a set of simple reflections generating W.

We will denote the face poset of G/P_J by Q^J.

In this case elements of Q^J are indexed by triples (x, u, w) where $x \in W^J_{max}, u \in W_J, w \in W^J_{min}$.

The Hasse diagram of the poset of cells of $(Gr_{2,4})_{\geq 0}$



A combinatorial result and a conjecture

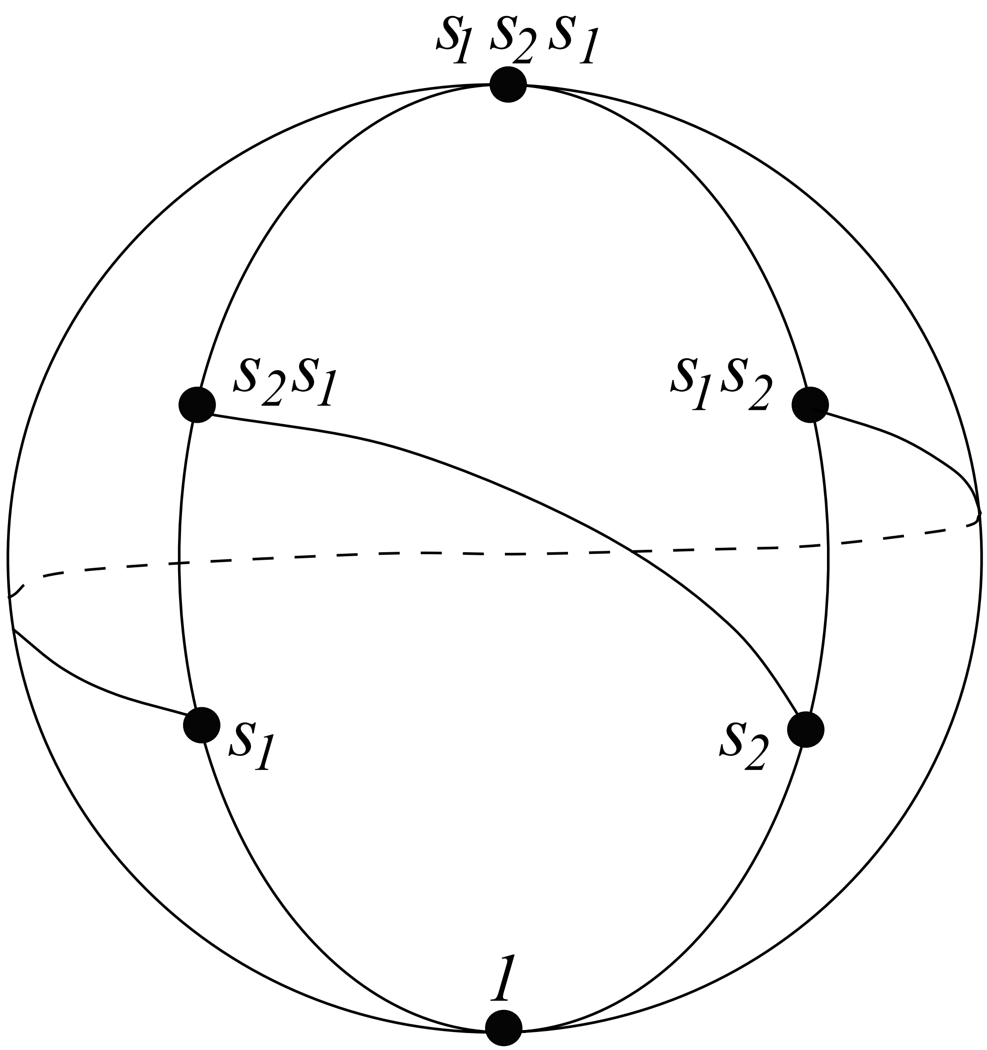
Given a cell complex K, we define its *face poset* as follows: the elements are the cells of K, and we say $\sigma < \tau$ if $\sigma \subset \overline{\tau}$.

Theorem. (W.) Let G be a (finite type) real reductive group and let G/P_J be any partial flag variety. Q^J – the face poset of $(G/P_J)_{\geq 0}$ – is the face poset of a regular CW complex homeomorphic to a ball.

Regular: the closure of each cell is a closed ball, and the closure of a cell minus the interior is a sphere.

CW complex: special kind of cell complex, where each cell σ comes with an *attaching map* – i.e. there is a map h from a closed ball B to K which is a homeomorphism from $\text{int}(B)$ to σ.

$(SL_3/B)_{\geq 0}$ with its cell decomposition



A combinatorial result and a conjecture – cont.

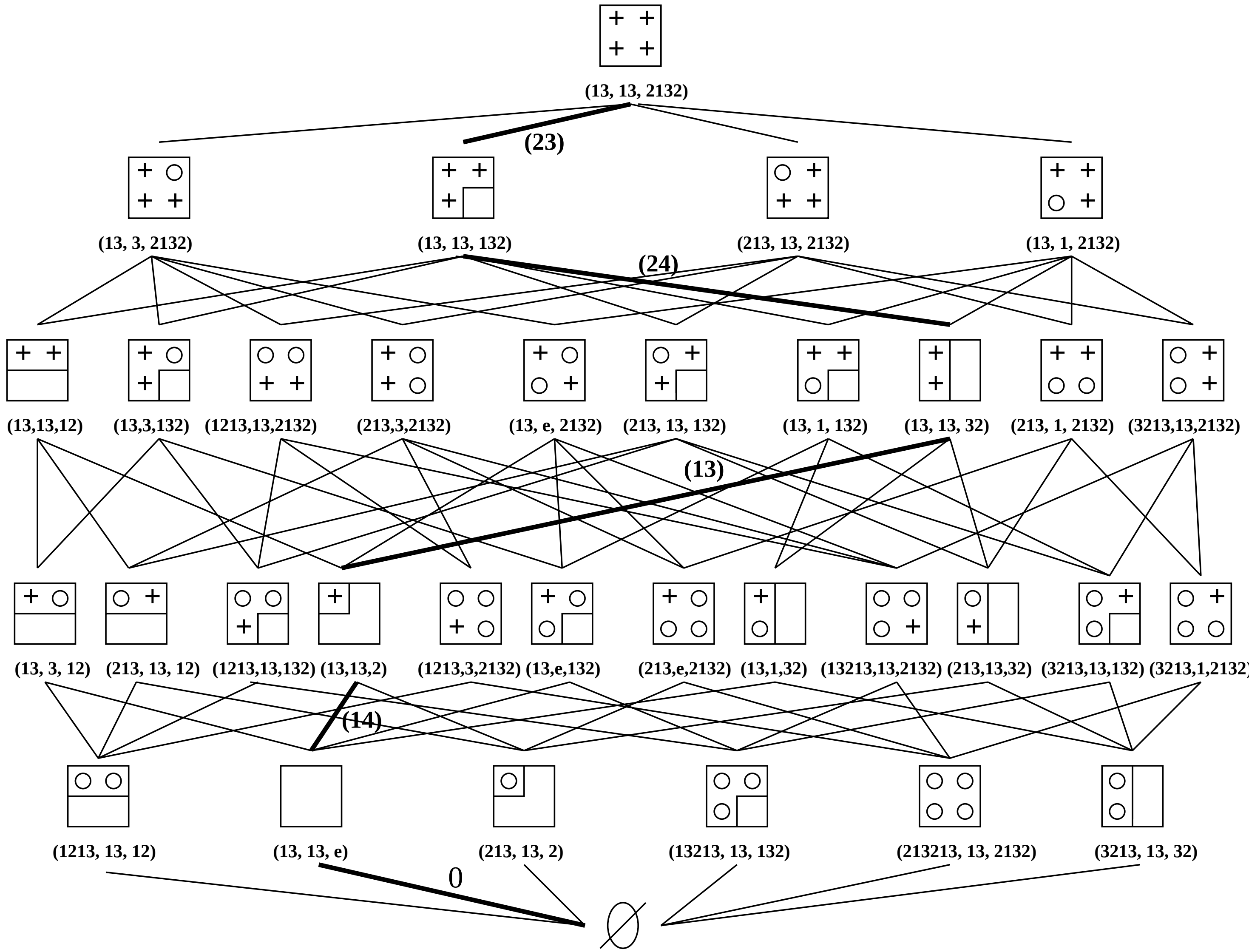
Theorem.(W.) Q^J – the face poset of $(G/P_J)_{\geq 0}$ – is the face poset of a regular CW complex homeomorphic to a ball.

Method of proof: finding a special labeling (an EL-labeling) of the edges of Q^J, proving that Q^J is *thin*, and using a result of Bjorner.

Conjecture. $(G/P_J)_{\geq 0}$ together with its cell decomposition is a regular CW complex homeomorphic to a ball.

First question: is this space actually a CW complex? Need to understand the cells better, and how they fit together.

The Hasse diagram of the poset of cells of $(Gr_{24})_{\geq 0}$



Parameterizations of cells

Postnikov has parameterizations of cells of $(Gr_{kn})_{\geq 0}$ in terms of certain *plabic graphs*.

Marsh-Rietsch have parameterizations of cells for any $(G/P)_{\geq 0}$.

Let $v \leq w$ and let $\mathbf{w} = (i_1, \ldots, i_m)$ encode a reduced expression $s_{i_1} \ldots s_{i_m}$ for w.

The *positive distinguished subexpression* for v in $\mathbf{w}$ is the "rightmost reduced subexpression" for v in $\mathbf{w}$.

Example (Type A, $s_i = (i\ i+1)$): $\mathbf{w} = s_2 s_1 s_3 s_2 s_3, v = s_3 s_1$.

We use the notation

$$\begin{aligned} \mathbf{v}_+ &:= \{j_1, \ldots, j_k\}, \\ \mathbf{v}_+^c &:= \{1, \ldots, m\} \setminus \{j_1, j_2, \ldots, j_k\}, \end{aligned}$$

when referring to this special subexpression for v in $\mathbf{w}$.

Parameterizations of cells – cont.

We define

$$\begin{aligned} \phi_{\mathbf{v}_+,\mathbf{w}} : (\mathbb{C}^*)^{\mathbf{v}_+^c} &\rightarrow \mathcal{R}_{v,w}, \\ (t_r)_{r\in\mathbf{v}_+^c} &\mapsto g_1 \dots g_m \cdot B^+, \end{aligned}$$

where

$$g_r = \begin{cases} \dot{s}_{i_r}, & \text{if } r \in \mathbf{v}_+, \\ y_{i_r}(t_r) & \text{if } r \in \mathbf{v}_+^c. \end{cases}$$

Theorem. (Marsh, Rietsch) The restriction of $\phi_{\mathbf{v}_+,\mathbf{w}}$ to $(\mathbb{R}_{>0})^{\mathbf{v}_+^c}$ defines an isomorphism of semi-algebraic sets,

$$\phi^{>0}_{\mathbf{v}_+,\mathbf{w}} : (\mathbb{R}_{>0})^{\mathbf{v}_+^c} \rightarrow \mathcal{R}^{>0}_{v,w}.$$

Example: $\mathbf{w} = s_2 s_1 s_3 s_2 s_3, v = s_3 s_1$.

Is $(G/P)_{\geq 0}$ a CW complex?

Example: $\mathbf{w} = s_2 s_1 s_3 s_2 s_3, v = s_1 s_3$.

$(t_1, t_2, t_3) \mapsto y_2(t_1)\dot{s}_1 y_3(t_2) y_2(t_3)\dot{s}_3 \cdot B^+$.

This gives a map from an *open* ball $(\mathbb{R}_{>0})^3$ to the cell $\mathcal{R}_{v,w}^{>0}$. But if we want to show $(G/P)_{\geq 0}$ is a CW complex, we need to extend this to a continuous map whose domain is a *closed* ball.

Problem: The behavior is complicated when parameters go to 0 or infinity: need to compactify $(\mathbb{R}_{>0})^{\mathbf{v}_+^c}$ and get a closed ball.

Idea: (Postnikov, Speyer, W.) Introduce an appropriate toric variety, and extend the parameterizing map to the non-negative part of the toric variety. The non-negative part of a toric variety is homeomorphic to a closed ball! (the moment polytope)

Toric varieties and their non-negative parts

Definition: Let $Q \subset \mathbb{R}^n$ be a lattice polytope and let $\{\mathbf{m}_i\}_{i=1}^{\ell}$ be the lattice points of $Q \cap \mathbb{Z}^n$. Consider the map $\phi : (\mathbb{C}^*)^n \to \mathbb{P}^{\ell-1}$ such that

$$(x_1, \dots, x_n) \mapsto [\mathbf{x}^{\mathbf{m}_1} : \cdots : \mathbf{x}^{\mathbf{m}_\ell}].$$

The associated *toric variety* is $X_Q := \overline{\operatorname{im} \phi}$.

The *real part* is $X_Q(\mathbb{R}) := X_Q \cap \mathbb{RP}^{\ell-1}$.

The *positive part* $(X_Q)_{>0}$ is the image of $(\mathbb{R}_{>0})^n$ under ϕ.

The *non-negative part* $(X_Q)_{\geq 0}$ is the closure in $X_Q(\mathbb{R})$ of $(X_Q)_{>0}$.

Fact:$(X_Q)_{\geq 0}$ is homeomorphic to Q (via the moment map).

CW structure

Lemma. (Postnikov, Speyer, W.) Suppose we have a map $\Phi : (\mathbb{R}_{>0})^n \to \mathbb{P}^{N-1}$ given by

$$(t_1, \dots, t_n) \mapsto [f_1(t_1, \dots, t_n), \dots, f_N(t_1, \dots, t_n)],$$

where the f_i's are polynomials with positive coefficients.
Let S be the set of all exponent vectors in $\mathbb{Z}^n$ which occur among the monomials of the f_i's, and let P be the convex hull of S.
Then Φ factors through the totally positive part of the toric variety $(X_P)_{>0}$, giving a map $h_{>0} : (X_P)_{>0} \to \mathbb{P}^{N-1}$. Moreover $h_{>0}$ extends continuously to the closure to give a well-defined map $h_{\geq 0} : (X_P)_{\geq 0} \to \overline{h_{>0}((X_P)_{>0})}$.

This is a way to construct an attaching map ... we extend a parameterization to the non-negative part of a toric variety, which is homeomorphic to a closed ball (its moment polytope)!

CW structure for the Grassmannian

For the Grassmannian, we can use Postnikov's parameterizations of cells in terms of *plabic graphs*.

Example:

Theorem. (Postnikov, Speyer, W.) The non-negative part of the Grassmannian is a CW complex. The polytopes which arise are planar analogs of Birkhoff polytopes, and can be described combinatorially in terms of "planar-perfect matchings" of plabic graphs.

Here everything is completely explicit: we can also describe vertices and facet inequalities of polytope in terms of plabic graphs. Connection to matroid polytopes.

CW structure for $(G/P)_{\geq 0}$ – the general case

We use the Marsh-Rietsch parameterizations of cells. We need to map each parameterization of a cell into projective space and show that this map is given by polynomials with positive coefficients.

When G is simply laced, consider the representation $V = V(\rho)$ of G with a highest weight vector η and canonical basis $\mathcal{B}(\rho)$.

Let $i : G/B \to \mathbb{P}(V)$ be the embedding which takes $g \cdot B_+$ to the line $\langle g \cdot \eta \rangle$. We specify points in the projective space $\mathbb{P}(V)$ using homogeneous coordinates corresponding to $\mathcal{B}(\rho)$.

One can use positivity properties of the canonical basis to show that the hypotheses of the PSW lemma are satisfied. Then use "descent" to get rid of simply-laced hypothesis.

Theorem. (Rietsch, W.) $(G/P)_{\geq 0}$ is a CW complex.

Topology of closures of cells

Theorem. (W.) (*Proof in progress*)
The closure of each cell of $(G/P)_{\geq 0}$ is contractible. In particular, we recover Lusztig's result that $(G/P)_{\geq 0}$ is contractible.

Main tool: discrete Morse theory.

Forman's Discrete Morse theory is a technique for analyzing the topology of a CW complex K. One finds a *discrete Morse function* f on the face poset of K. Under certain hypotheses, K is then homotopy equivalent to a CW complex with fewer cells in each dimension (the number is determined by the *critical cells* of f).

In some situations, a discrete Morse function on K is equivalent to a *Morse matching* on the face poset of K.

What is a Morse matching?

Consider the Hasse diagram of the face poset of K. (There is a point for each cell, and edges illustrate *minimal* order relations.)

Given a *matching* M of the edges, we associate a directed graph to M as follows: direct edges in M up (from lower to higher-dimensional cells), and all other edges down. M is called a *Morse matching* if the resulting graph is acyclic.

A cell is called *critical* with respect to M if it is not incident to an edge of M.

The topology of closures of cells

Theorem. Let K be a CW complex with the *subcomplex property*, whose face poset Q is *sturdy*. Suppose that Q has a *Morse matching* M, such that whenever $(\sigma^{(p)}, \tau^{(p+1)}) \in M$, σ is a *regular* face of τ. Let $m_p(M)$ denote the number of critical cells of dimension p. Then K is homotopy equivalent to a CW complex with $m_p(M)$ cells of dimension p.

Main idea of proof of contractibility result: For the face poset of the closure of each cell, we want to find a Morse matching with only one critical cell of dimension 0 – this will prove contractibility. Also need to make sure that whenever (σ, τ) are matched, σ is a regular face of τ.

The topology of closures of cells

- Some pairs of cells are easier to analyze than others. E.g. if τ has a parameterization by $(t_1, \ldots, t_p)$ such that setting a $t_i = 0$ gives a parameterization of another cell σ, then the polytope P' for σ is a *facet* of the polytope P for τ – obtained by intersecting P with a coordinate hyperplane.

- Even in this situation it's tricky to verify regularity. The polytopes are not very explicit because they're defined using the canonical basis. Helpful (nontrivial) fact: the origin is a vertex of each such polytope.

- Need to find a Morse matching whose matched edges correspond to cells (σ, τ) of the type above.

The topology of closures of cells

• Show that one can go from a CL-labeling of our poset to a perfect Morse matching – a Morse matching with only one critical cell of dimension 0.

• Find the appropriate CL-labeling such that the resulting Morse matching will have matched edges as desired. To do so, we use ideas of Bjorner-Wachs (who found a CL-labeling of Bruhat order using reduced expressions), and also ideas of Dyer (who found an EL-labeling of Bruhat order using reflection orders).

• Then Morse theory machinery gives a "collapsing" of the closure of each cell of $(G/P)_{\geq 0}$, which implies contractibility.

• This generalizes Lusztig's result that the closure $(G/P)_{\geq 0}$ of the top cell $(G/P)_{>0}$ is contractible. Gives some justification for his "remarkable polyhedral subspace" comment.

Thank you!

Preprints available at:

- http://www.math.harvard.edu/~lauren

Shelling totally nonnegative flag varieties, Crelle '07.

Matching polytopes, toric geometry, and the non-negative part of the Grassmannian (with A. Postnikov and D. Speyer), arXiv:0706.2501.

The non-negative part of G/P is a CW complex (with K. Rietsch), in preparation.

The topology of $(G/P)_{\geq 0}$ and its cell decomposition, in preparation.