- $\mathfrak{g}$ simple complex Lie algebra of rank n (simply laced)
- $\mathfrak{g} = \mathfrak{n}^+ \otimes \mathfrak{h} \otimes \mathfrak{n}^-$

loop algebra $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, $[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$

current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$

- $\mathfrak{h}$ Cartan subalgebra, α a root, α_i simple root, R set of roots, h_i simple coroot, $x_i^{\pm}$ simple root vectors
- λ weight, ω_i fundamental weight, P weights, P^+ dominant weights, $P^{\vee}$ coweights
- $\hat{\mathfrak{g}}$ the affine Kac Moody algebra corresponding to $\mathfrak{g}$
- $\hat{\mathfrak{g}} = \hat{\mathfrak{b}} \oplus \hat{\mathfrak{n}_-} = \hat{\mathfrak{n}_+} \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}_-}$, $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$
- Λ weight for $\hat{\mathfrak{g}}$, Λ_0 the fundamental weight of the basic representation
- W Weyl group of $\mathfrak{g}$, $W^{aff} = W \ltimes Q^{\vee}$, $\tilde{W} = W \ltimes P^{\vee} = \Sigma \ltimes W^{aff}$
- σ a non-trivial diagram automorphism of $\mathfrak{g}$ of order $m(\in \{2, 3\})$.
- let ζ be a primitive m-th root of unity
- extend to $L(\mathfrak{g})$ by $\sigma(x \otimes t^r) = \zeta^r \sigma(x) \otimes t^r$
- $\mathfrak{g}_0$, $L^{\sigma}(\mathfrak{g})$, $\mathfrak{g}^{\sigma}[t]$, the fix points
- $\mathfrak{g}_0$ simple Lie algebra, $\mathfrak{g}_{\epsilon}$ irred. repr. of $\mathfrak{g}_0$ (θ_s or $2\theta_s$)
- I_0, R_0, P_0
- $x_{i,0} = x_i$ if $\sigma(i) = i$, else $x_{i,0} = x_i + x_{\sigma(i)}$. $x_{i,1} = x_i - x_{\sigma(i)}$
- $A_{2n}^{(2)}$ and $i = n$ different

The monoid $\mathcal{P}^+$

Let $\mathcal{P}^+$ be the monoid of I–tuples of polynomials $\pi_i = (\pi_1, \cdots, \pi_n)$ in an indeterminate u with constant term one, with multiplication being defined component wise. For $i \in I$ and $a \in \mathbb{C}^\times$, set

$$\pi_{i,a} = ((1 - au)^{\delta_{ij}} : j \in I) \in \mathcal{P}^+, \tag{1}$$

and for $\lambda \in P^+$, set

$$\pi_{\lambda,a} = \prod_{i \in I} (\pi_{i,a})^{\lambda(h_i)}, \quad \lambda \neq 0.$$

Clearly any $\pi^+ \in \mathcal{P}^+$ can be written uniquely as a product

$$\pi^+ = \prod_{k=1}^{\ell} \pi_{\lambda_i, a_i},$$

for some $\lambda_1, \cdots, \lambda_\ell \in P^+$ and distinct elements $a_1, \cdots, a_\ell \in \mathbb{C}^\times$. Define a map $\mathcal{P}^+ \to P^+$ by $\pi \to \lambda_\pi = \sum_{i \in I} \deg(\pi_i)\omega_i$.

The modules $W(\pi)$, $V(\pi)$.
Given $\pi = (\pi_i)_{i\in I} \in \mathcal{P}^+$ with $\pi = \prod_{k=1}^{\ell} \pi_{\lambda_\ell, a_\ell}$ where $a_1, \cdots, a_\ell$ are all distinct, let $W(\pi)$ be the $L(\mathfrak{g})$–module generated by an element w_π with relations:

$$L(\mathfrak{n}^+)w_\pi = 0, \quad hw_\pi = \lambda_\pi(h)w_\pi, \quad (x_i^-)^{\lambda_\pi(h_i)+1}w_\pi = 0,$$
$$(h \otimes t^r)w_\pi = \left(\sum_{j=1}^{\ell} \lambda_j(h)a_j^r\right) w_\pi.$$

$i \in I$ and $h \in \mathfrak{h}$.

Let $b \in \mathbb{C}^\times$ and let $\tau_b W(\pi)$ be the $L(\mathfrak{g})$–module obtained by pulling back $W(\pi)$ through the automorphism τ_b of $L(\mathfrak{g})$, where $\tau_b(x \otimes t^r)b^r x \otimes t^r$.

Lemma 1. *1. Let $\pi \in \mathcal{P}^+$. Then $W(\pi) = \mathbf{U}(L(\mathfrak{n}^-))w_\pi$, and hence we have,*

$$\mathbf{wt}(W(\pi)) \subset \lambda_\pi - Q^+, \quad \dim W(\pi)_{\lambda_\pi} = 1.$$

In particular, the module $W(\pi)$ has a unique irreducible quotient $V(\pi)$.

2. For $b \in \mathbb{C}^\times$, we have $\tau_b W(\pi) \cong W(\pi_b)$, where $\pi = (\pi_i(u))_{i\in I}$ and $\pi_b = (\pi_i(b^{-1}u))_{i\in I}$. In particular we have

$$W(\pi_{\lambda,a}) \cong_{\mathfrak{g}} W(\pi_{\lambda,ab}).$$

In particular, the irreducible modules are exacly the tensor products of evaluation modules.

Some results on Weyl modules

Theorem 1. *(i) Given $\pi = (\pi_i)_{i\in I}$ with unique decomposition $\pi = \prod_{k=1}^{\ell} \pi_{\lambda_\ell, a_\ell}$, we have an isomorphism of $L(\mathfrak{g})$–modules*

$$W(\pi) \cong \otimes_{k=1}^{\ell} W(\pi_{\lambda_k, a_k}).$$

(ii) Let V be any finite–dimensional $L(\mathfrak{g})$–module generated by an element $v \in V$ such that

$$L(\mathfrak{n}^+)v = 0, \quad L(\mathfrak{h})v = \mathbb{C}v.$$

Then there exists $\pi \in \mathcal{P}^+$ such that the assignment $w_\pi \to v$ extends to a surjective homomorphism $W(\pi) \to V$ of $L(\mathfrak{g})$–modules.

(iii) Let $\lambda \in P^+$ and $a \in \mathbb{C}^\times$. Suppose that $\lambda = \sum_{i\in I} m_i\omega_i$. Then

$$W(\pi_{\lambda,a}) \cong_{\mathfrak{g}} \bigotimes_{i\in I} W(\pi_{\omega_i,1})^{\otimes m_i}.$$

□

From untwisted to twisted

Let $(\ ,\ )$ be the form on $\mathfrak{h}_0^*$ induced by the Killing form of $\mathfrak{g}_0$ normalized so that $(\theta_0,\theta_0)=2$. For $i\in I_0$ and $a\in\mathbb{C}^\times$, $\lambda\in P_0^+$ and $\mathfrak{g}$ not of type A_{2n} let

$$\pi^\sigma_{i,a}=((1-a^{(\alpha_i,\alpha_i)}u)^{\delta_{ij}}:j\in I_0),\qquad \pi^\sigma_{\lambda,a}=\prod_{i\in I_0}\left(\pi^\sigma_{i,a}\right)^{\lambda(h_i)},$$

while if $\mathfrak{g}$ is of type A_{2n} we set for $i\in I_0$, $a\in\mathbb{C}^\times$, $\lambda\in P_\sigma^+$,

$$\pi^\sigma_{i,a}=((1-au)^{\delta_{ij}}:j\in I_0),\qquad \pi^\sigma_{\lambda,a}=\prod_{i\in I_0}\left(\pi^\sigma_{i,a}\right)^{(1-\frac{1}{2}\delta_{i,n})\lambda(h_i)}.$$

Let $\mathcal{P}_\sigma^+$ be the monoid generated by the elements $\pi^\sigma_{\lambda,a}$. Define a map $\mathcal{P}_\sigma^+\to\mathcal{P}_{\prime}^+$ by

$$\lambda_{\pi^\sigma}=\sum_{i\in I_0}(\deg\pi_i^\sigma)\omega_i,$$

if $\mathfrak{g}$ is not of type A_{2n} and

$$\lambda_{\pi^\sigma}=\sum_{i\in I_0}(1+\delta_{i,n})(\deg\pi_i^\sigma)\omega_i,$$

if $\mathfrak{g}$ is of type A_{2n}. It is clear that any $\pi^\sigma\in\mathcal{P}_\sigma^+$ can be written (non–uniquely) as product

$$\pi^\sigma=\prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\pi^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k},$$

where $\mathbf{a}=(a_1,\cdots,a_\ell)$ and $\mathbf{a}^m$ have distinct coordinates. We call any such expression a standard decomposition of π^σ.

Given $\lambda = \sum_{i\in I} m_i\omega_i \in P^+$ and $0 \le \epsilon \le m-1$, define elements $\lambda(\epsilon) \in P_\sigma^+$ by,

$$\lambda(0) = \sum_{i\in I_0} m_i\omega_i, \quad \lambda(1) = \sum_{i\in I_0:\sigma(i)\neq i} m_{\sigma(i)}\omega_i,$$

$$\text{if } \ m=2 \ \text{ and } \ \mathfrak{g} \text{ not of type } A_{2n}$$

$$\lambda(0) = \sum_{i\in I_0}(1+\delta_{i,n})m_i\omega_i, \quad \lambda(1) = \sum_{i\in I_0:\sigma(i)\neq i}(1+\delta_{\sigma(i),n})m_{\sigma(i)}\omega_i,$$

$$\text{if } \ m=2 \ \text{ and } \ \mathfrak{g} \text{ of type } A_{2n}$$

$$\lambda(0) = m_1\omega_1 + m_2\omega_2, \quad \lambda(1) = m_3\omega_1, \quad \lambda(2) = m_4\omega_1, \quad \text{if } \ m=3.$$

Define a map $\mathbf{r} : \mathcal{P}^+ \to \mathcal{P}_\sigma^+$ as follows. Given $\pi \in \mathcal{P}^+$ write

$$\pi = \prod_{k=1}^{\ell} \pi_{\lambda_k,a_k}, \quad a_k \neq a_p, \quad 1 \le k \neq p \le \ell,$$

and set

$$\mathbf{r}(\pi) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1} \pi^\sigma_{\lambda_k(\epsilon),\zeta^\epsilon a_k}.$$

Note that $\mathbf{r}$ is well defined since the choice of (λ_k, a_k) is unique and set

$$\mathbf{i}(\pi^\sigma) = \{\pi \in \mathcal{P}^+ : \mathbf{r}(\pi) = \pi^\sigma\}.$$

The set $\mathbf{i}(\pi^\sigma)$

Lemma 2. *1. Let $i \in I_0$ and $a \in \mathbb{C}^\times$. We have,*

$$\mathbf{i}(\pi^\sigma_{\omega_i,a}) = \{\pi_{\sigma^r(\omega_i),\zeta^{m-r}a} \,|\, 0 \le r < m\},$$

and for A^2_{2n} and $i = n$,

$$\mathbf{i}(\pi^\sigma_{2\omega_n,a}) = \{\pi_{\omega_n,a}\,,\pi_{\omega_{n+1},-a}\}$$

2. Let $\pi^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}(\pi^\sigma_{\omega_i,\zeta^\epsilon a_k})^{m_{k,\epsilon,i}}$ be a decomposition of π^σ into linear factors for $\mathfrak{g}$ not of type A_{2n}. Then

$$\mathbf{i}(\pi^\sigma) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}\{\pi_{\sigma^r(\omega_i),\zeta^{m-r+\epsilon}a_k} \,|\, 0 \le r < m\}^{m_{k,\epsilon,i}}$$

where the product of the sets is understood to be the set of products of elements of the sets.
In the case of $A^{(2)}_{2n}$, let $\pi^\sigma = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{1}\prod_{i\in I_0}(\pi^\sigma_{(1+\delta_{i,n})\omega_i,\zeta^\epsilon a_k})^{m_{k,\epsilon,i}}$ be a decomposition of π^σ into linear factors. Then

$$\mathbf{i}(\pi^\sigma) = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{2}\prod_{i\in I_0}\{\pi_{\sigma^r(\omega_i),\zeta^{2-r+\epsilon}a_k} \,|\, 0 \le r < 2\}^{m_{k,\epsilon,i}}$$

3. In particular, we have

$$\prod_{k=1}^{\ell}\pi_{\mu_k,a_k} = \prod_{k=1}^{\ell}\prod_{\epsilon=0}^{m-1}\prod_{i\in I_0}\pi^{m_{k,\epsilon,i}}_{\sigma^\epsilon(\omega_i),a_k} \in \mathbf{i}(\pi^\sigma),$$

where $\mu_k = \sum_{\epsilon=0}^{m-1}\sum_{i\in I_0} m_{k,\epsilon,i}\sigma^\epsilon(\omega_i)$ and $a_i^m \ne a_j^m$.

The modules $W(\pi^\sigma)$, $V(\pi^\sigma)$
Given $\pi^\sigma = (\pi_{i,\sigma})_{i\in I_0} \in \mathcal{P}_\sigma^+$, with If $\pi^\sigma = \prod_{k=1}^{\ell} \pi^\sigma_{\lambda_k,a_k} \in \mathcal{P}_\sigma^+$,the Weyl module $W(\pi^\sigma)$ is the $\mathbf{U}(L^\sigma(\mathfrak{g}))$-module generated by an element w_{π^σ} with relations:

$$L^\sigma(\mathfrak{n}^+)w_{\pi^\sigma} = 0, \quad hw_\pi = \lambda_\pi(h)w_{\pi^\sigma}, \quad (x_{i,0}^-)^{\lambda_\pi(h_{i,0})+1}w_{\pi^\sigma} = 0,$$

And if $\mathfrak{g}$ not of type A_{2n},

$$(h_{i,\epsilon} \otimes t^{mk-\epsilon})w_{\pi^\sigma} = \sum_{j=1}^{\ell} \lambda_j(h_{i,0})a_j^{mk-\epsilon}w_{\pi^\sigma}, \tag{2}$$

and for $\mathfrak{g}$ of type A_{2n},

$$(h_{i,\epsilon} \otimes t^{mk-\epsilon})w_{\pi^\sigma} = \sum_{j=1}^{\ell}(1 - \frac{1}{2}\delta_{i,n})\lambda_j(h_{i,0})a_j^{mk-\epsilon}w_{\pi^\sigma}. \tag{3}$$

for all $i \in I_0$ and $h \in \mathfrak{h}_0$.

For $b \in \mathbb{C}^\times$ we have $\tau_b(L^\sigma(\mathfrak{g})) \subset L^\sigma(\mathfrak{g})$ and we let $\tau_b W(\pi^\sigma)$ be the $L^\sigma(\mathfrak{g})$–module obtained by pulling back $W(\pi^\sigma)$ through τ_b.

Lemma 3. 1. *Let $\pi^\sigma \in \mathcal{P}_\sigma^+$. Then $W(\pi^\sigma) = \mathbf{U}(L^\sigma(\mathfrak{n}^-))w_\pi^\sigma$, and hence we have,*

$$\mathbf{wt}(W(\pi^\sigma)) \subset \lambda_{\pi^\sigma} - Q_0^+, \qquad \dim W(\pi^\sigma)_{\lambda_{\pi^\sigma}} = 1.$$

In particular, the module $W(\pi^\sigma)$ has a unique irreducible quotient $V(\pi^\sigma)$.

2. *For $b \in \mathbb{C}^\times$, we have $\tau_b W(\pi^\sigma) \cong W(\pi_b^\sigma)$, where $\pi^\sigma = (\pi_i(u))_{i\in I}$ and $\pi_b^\sigma = (\pi_i(b^{-1}u))_{i\in I}$. In particular we have*

$$W(\pi^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} W(\pi^\sigma_{\lambda,ba}).$$

□

Some results for twisted Weyl modules

Theorem 2. *1. Let $\pi^\sigma \in \mathcal{P}_\sigma^+$. For all $\pi \in \mathbf{i}(\pi^\sigma)$, we have*

$$W(\pi^\sigma) \cong_{L^\sigma(\mathfrak{g})} W(\pi), \qquad V(\pi^\sigma) \cong_{L^\sigma(\mathfrak{g})} V(\pi).$$

2. Let $\pi^\sigma \in \mathcal{P}_\sigma^+$ and assume that $\prod_{k=1}^{\ell} \prod_{\epsilon=0}^{m-1} \pi^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k} \in \mathcal{P}_\sigma^+$ is a standard decomposition of π. As $L^\sigma(\mathfrak{g})$–modules, we have

$$W(\pi^\sigma) \cong \bigotimes_{k=1}^{\ell} W(\prod_{\epsilon=0}^{m-1} \pi^\sigma_{\lambda_{k,\epsilon},\zeta^\epsilon a_k}).$$

3. Suppose that $\prod_{\epsilon=0}^{m-1} \pi^\sigma_{\lambda_\epsilon,\zeta^\epsilon a} \in \mathcal{P}_\sigma^+$. Then

$$W(\prod_{\epsilon=0}^{m-1} \pi^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}) \cong_{\mathfrak{g}_0} \bigotimes_{\epsilon=0}^{m-1} W(\pi^\sigma_{\lambda_\epsilon,\zeta^\epsilon a}).$$

4. Let $\lambda = \sum_{i\in I_0} m_i\omega_i \in P_\sigma^+$ and $a \in \mathbb{C}^\times$. We have for $\mathfrak{g}$ not of type A_{2n}

$$W(\pi^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} \bigotimes_{i=1}^{n} W(\pi^\sigma_{\omega,1})^{\otimes m_i}$$

and for $\mathfrak{g}$ of type A_{2n}

$$W(\pi^\sigma_{\lambda,a}) \cong_{\mathfrak{g}_0} W(\pi^\sigma_{2\omega_n,1})^{\otimes \frac{m_n}{2}} \otimes \bigotimes_{i=1}^{n-1} W(\pi^\sigma_{\omega_i,1})^{\otimes m_i}.$$

5. Let V be any finite–dimensional $L^\sigma(\mathfrak{g})$–module generated by an element $v \in V$ such that

$$L^\sigma(\mathfrak{n}^+)v = 0, \quad L^\sigma(\mathfrak{h})v = \mathbb{C}v.$$

Then there exists $\pi^\sigma \in \mathcal{P}_\sigma^+$ *such that the assignment* $w_{\pi^\sigma} \to v$ *extends to a surjective homomorphism* $W(\pi^\sigma) \to V$ *of* $L^\sigma(\mathfrak{g})$*–modules.*

$\square$