1/25/08

Applications of Dual Equivalence Graphs

Sami Assaf

These notes are meant to augment the slides prepared by Sami Assaf.

Applications to symmetric functions, etc.

Dual equivalence graph is a graph on standard tableaux. they are modeled after crystal graphs.

SSYT (λ) = the set of semistandard young tableaux of partition type λ.

You can describe the connections completely combinatorally

Question: If we create a directed graph and color the edges, when do we get one of these graphs??

---

the elementary dual equivalence ~~the~~ is dual to Knuth relations

---

+'s and −'s have to do w/ order of fillings i.e. if 1 comes before 2 get a +, otherwise a −.

the labeled edges (i.e $\xrightarrow{i}$) are <u>not</u> involutions. So we never get the same labeled edge coming out of a tableaux more than once..

Given a generating function, if its symmetric ... how can we write it in terms of Schur functions

If instead we have our objects endowed w/ a statistic. Then we can write an analogous formula if the statistic stays constant on components.

inv_k = the k^{th} inversion #.

the LLT polynomials <u>are</u> symmetric
(they proved it using Fock spaces)

| 1 | 1 | 1 | | 2 | 2 | 2 | —can only go to→ | 2 | 2 | 2 | | 1 | 1 | 2 |

(no other choice)

DEG requires standard k-tuple of tableaux
rather than k-tuple of standard tableaux.

Once we define the involution $D_i^{(k)}(T)$

it is easy to show that we get a DEG in the case when $k = 2$.
(unfortunately it doesn't work in general).

DEG corresponds to a single Schur function

D-graph corresponds to a sum of Schur functions.

There is an explicit algorithm for changing any D-graph into several DEG's.

---

maj(S) is some major ~~[illegible]~~ index.
it turns out
Macdonald poly's can be written as sums of LLT poly's.

This proves that Macdonald polynomials are Schur positive.

Applications of dual equivalence graphs

Sami Assaf

University of Pennsylvania

sassaf@math.mit.edu

Outline

I. Graphs on tableaux
- Crystal graphs
- Dual equivalence graphs

II. Symmetric functions
- Schur positivity from graphs
- LLT and Macdonald polynomials

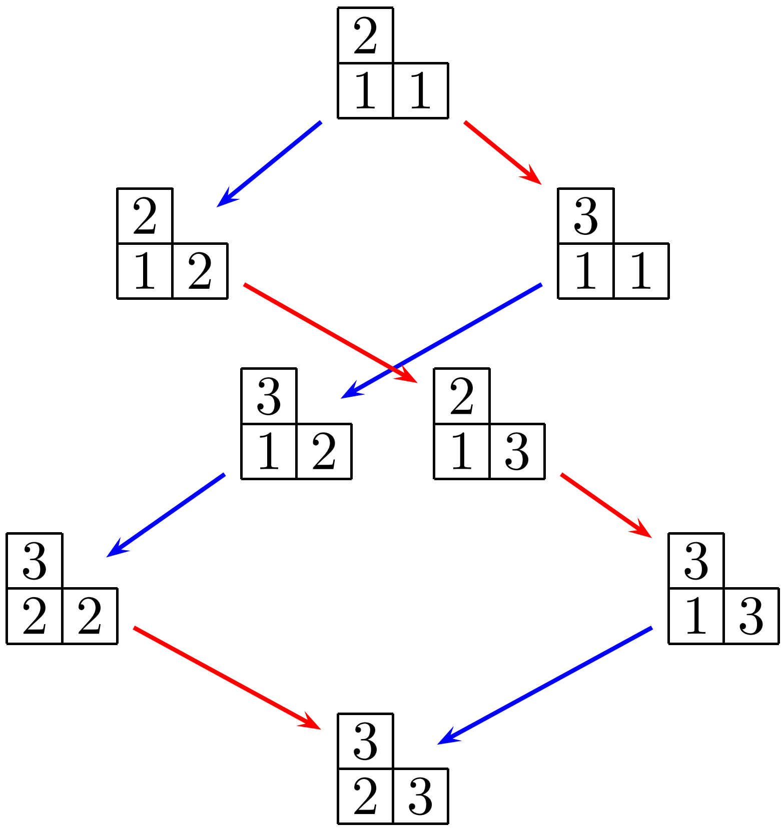
III. Representation Theory
- Weyl groups acting on 0-weight spaces
- From crystal graphs to dual equivalence graphs

I. Graphs on Tableaux

Crystal graphs

In 1990, Kashiwara introduced crystal bases for representations of quantized universal enveloping algebras at $q = 0$.

A *crystal graph* is a directed, colored graph on the crystal basis with edges given by deformations of the Chavelley generators e_i and f_i.

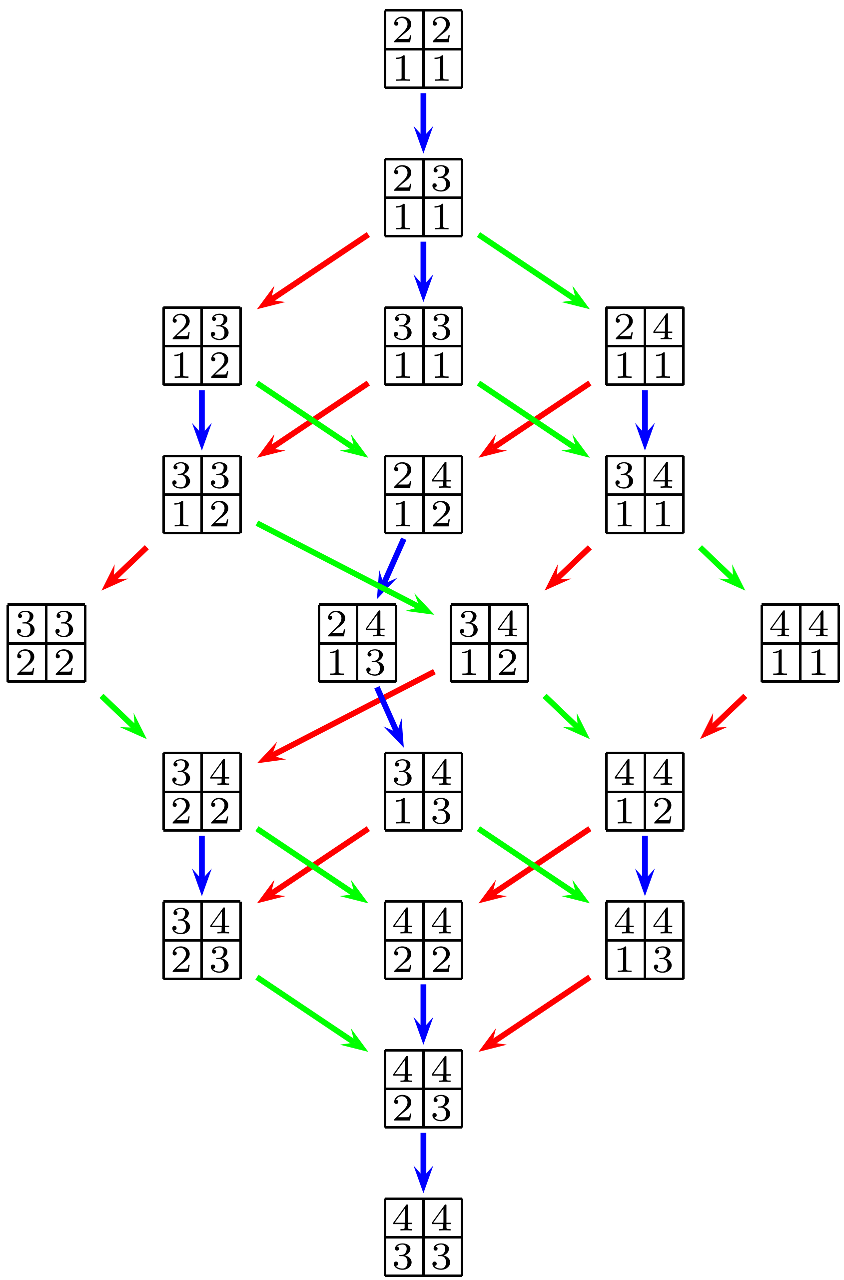


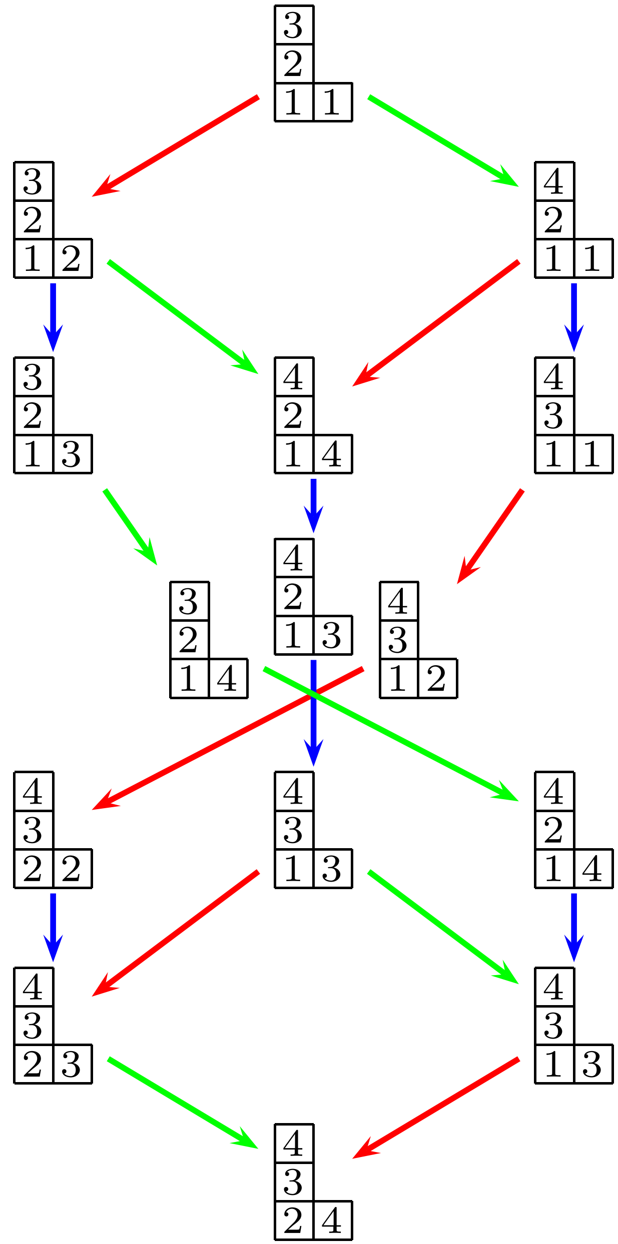
Let E_λ be the irreducible representation of $U_q(\mathfrak{sl}_n)$ with highest weight λ.

The crystal basis for E_λ is indexed by $\mathrm{SSYT}(\lambda)$. Denote the crystal graph by $\mathcal{X}_\lambda^n$.

Littelmann and Kashiwara-Nakashima gave combinatorial descriptions of crystal graphs in terms of SSYT.

Examples of crystal graphs

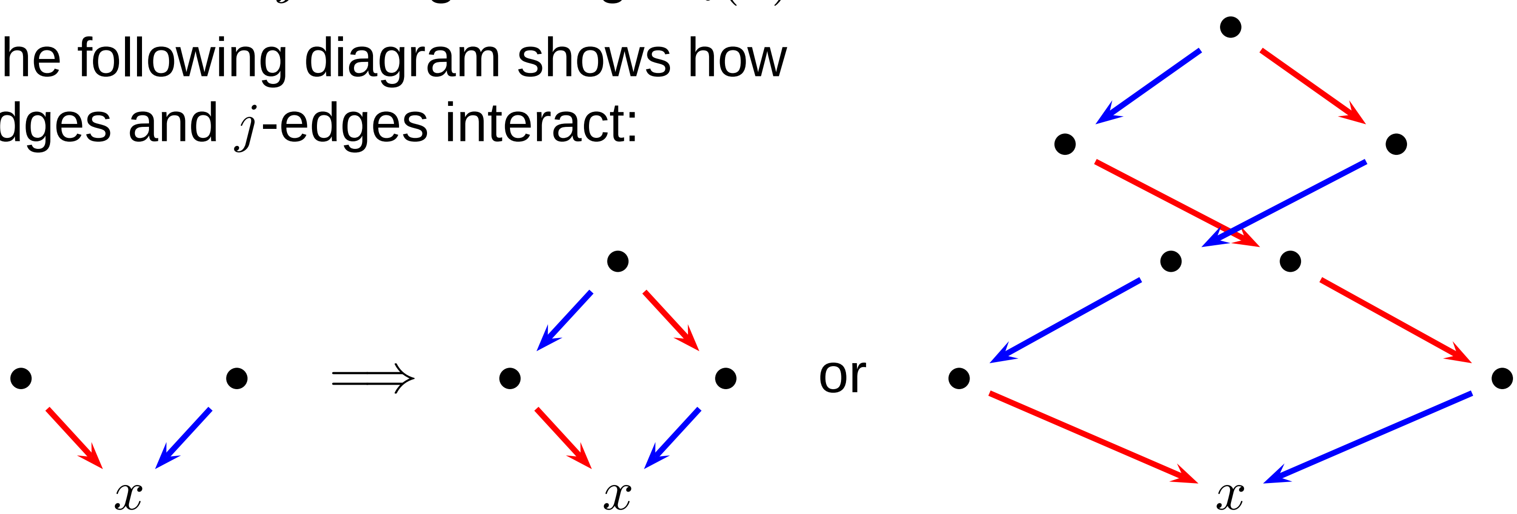




Local characterization of crystal graphs

Stembridge gave a local characterization for type A crystals:

- All monochromatic directed paths have finite length.
- For every x, there is at most one edge $x \xrightarrow{i} y$ (resp. $x \xleftarrow{i} z$).
- The j-string through x is either 1 shorter at the tail or 1 longer at the head than the j-string through $\widetilde{e}_i(x)$.
- The following diagram shows how i-edges and j-edges interact:



Theorem. (Stembridge 2003) The graph $\mathcal{X}^n_\lambda$ satisfies the above axioms, and any connected component of a directed, colored graph satisfying the axioms is isomorphic to $\mathcal{X}^n_\lambda$ for some λ, n.

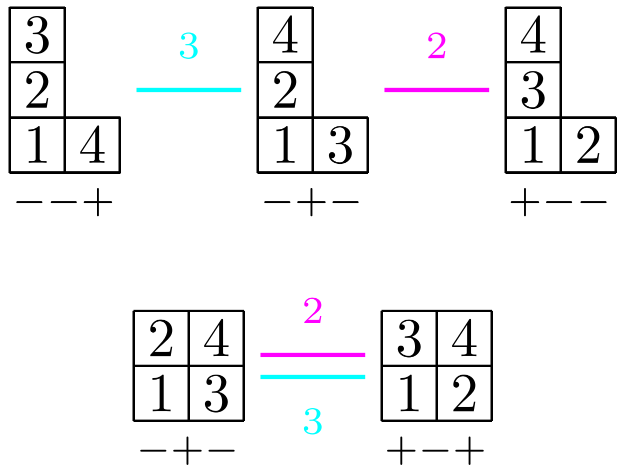
Dual equivalence graphs

Motivated by crystal graphs, in 2005, Haiman suggested defining a graph on standard tableaux using the dual equivalence relation.

An *elementary dual equivalence* for $i-1, i, i+1$ is

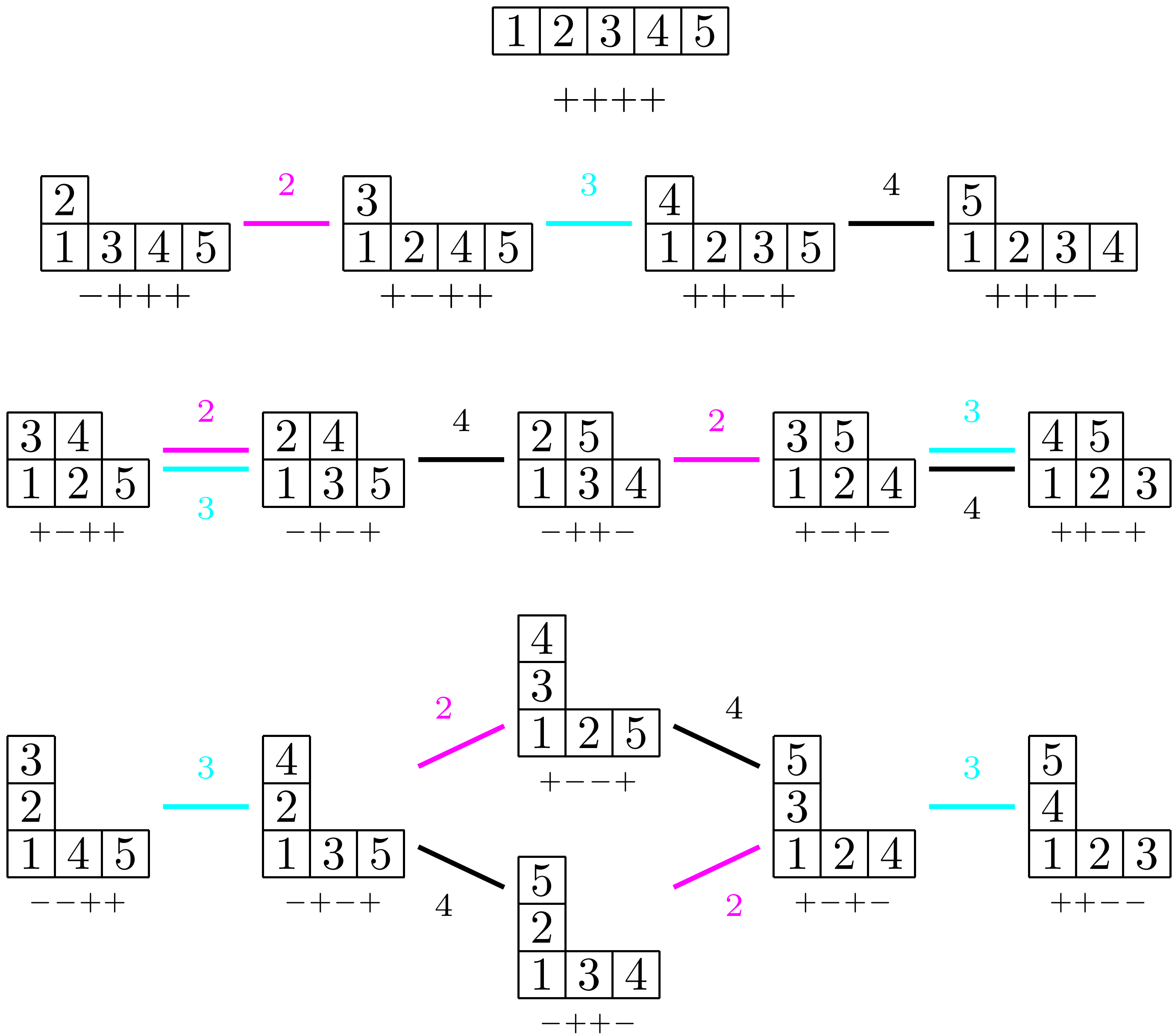
$$\cdots\ i\ \cdots\ i \pm 1\ \cdots\ i \mp 1\ \cdots\ \equiv^*\ \cdots\ i \mp 1\ \cdots\ i \pm 1\ \cdots\ i\ \cdots$$

The graph $\mathcal{G}_\lambda$ is the (vertex-signed) edge-colored graph on $\mathrm{SYT}(\lambda)$ with an i-edge between two tableaux which differ by an elementary dual equivalence for $i-1, i, i+1$.



Here $\sigma(T) \in \{\pm\}^{n-1}$ is given by $\sigma(T)_i \;=\; \begin{cases} + & \text{if } i \text{ left of } i+1 \text{ in } w_T \\ - & \text{if } i+1 \text{ left of } i \text{ in } w_T \end{cases}$

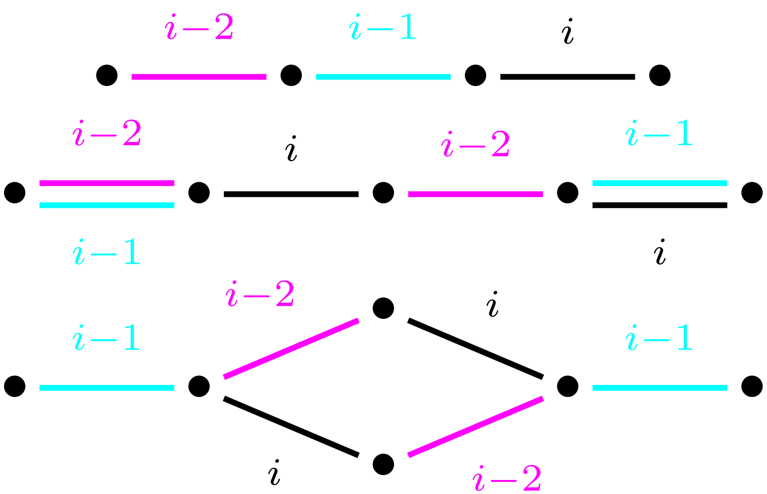
Examples of DEGs



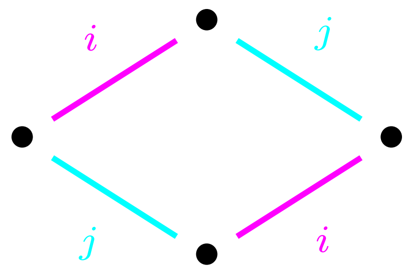
A local characterization for DEGs

Definition. (A. 2007) A *dual equivalence graph* (DEG) is a vertex-signed, edge-colored graph satisfying the following:

- across an i-edge, $\cdots ?\ +\ -\ ?\ \cdots \overset{i}{\text{———}} \cdots ?\ -\ +\ ?\ \cdots$
- if $\cdots\ -\ +\ -\ \cdots \overset{i}{\text{———}} \cdots\ +\ -\ +\ \cdots$, then $\bullet \underset{i-1}{\overset{i}{=\!=\!=}} \bullet$
- For 3 consecutive edge colors, components of the graph are



- For $|i - j| \geq 3$,



Theorem. (A. 2007) The graph $\mathcal{G}_\lambda$ is a DEG and every connected component of a DEG is isomorphic to $\mathcal{G}_\lambda$ for a unique partition λ.

II. Symmetric Functions

Combinatorially, the Schur function s_λ may be defined by

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^{\mathrm{wt}(T)} = \sum_{T \in \mathrm{SYT}(\lambda)} Q_{\sigma(T)}(\mathbf{x}).$$

Given a generating function for some set of (standard) objects,

$$\sum_{S \in \mathrm{objects}} q^{\mathrm{stat}(S)} \mathbf{x}^{\mathrm{wt}(S)} = \sum_{S \in \substack{\mathrm{standard} \\ \mathrm{objects}}} q^{\mathrm{stat}(S)} Q_{\sigma(S)}(\mathbf{x})$$

↓ construct a crystal graph $\mathcal{X}$ on objects ↓

$$\sum_\lambda \left(\sum_{\mathcal{C} \cong \mathcal{X}^n_\lambda} q^{\mathrm{stat}(\mathcal{C})} \right) s_\lambda(\mathbf{x})$$

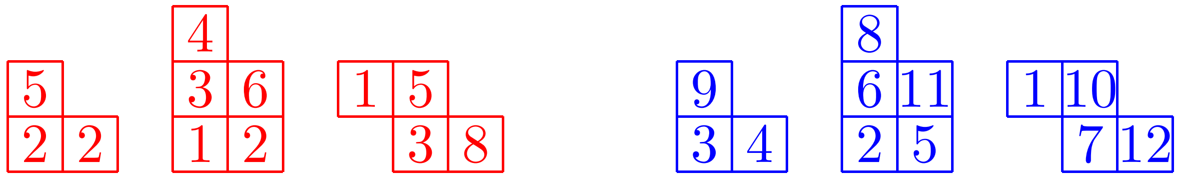
↓ construct a DEG $\mathcal{G}$ on standard objects ↓

$$\sum_\lambda \left(\sum_{\mathcal{C} \cong \mathcal{G}_\lambda} q^{\mathrm{stat}(\mathcal{C})} \right) s_\lambda(\mathbf{x})$$

where stat is *constant on connected components* of the graph.

In 1997, Lascoux, Leclerc and Thibon (LLT) defined $\widetilde{G}_{\mu}^{(k)}$ to be the generating function for k-tuples of tableaux weighted by inv_k:

$$\widetilde{G}_{\mu}^{(k)}(\mathbf{x};q) = \sum_{\mathbf{T}\in \mathrm{SSYT}_k(\mu)} q^{\mathrm{inv}_k(\mathbf{T})}\mathbf{x}^{\mathrm{wt}(\mathbf{T})} = \sum_{\mathbf{T}\in \mathrm{SYT}_k(\mu)} q^{\mathrm{inv}_k(\mathbf{T})} Q_{\sigma(\mathbf{T})}(\mathbf{x}).$$



In 2005, van Leeuwen (building on work of Carré-Leclerc) gave a crystal-theoretic proof for $k=2$ taking approximately 50 pages.

A DEG structure is far simpler: use dual equivalence whenever this preserves inv_k, and otherwise does the only other "local" move.

A graph for LLT polynomials

The key to defining edges lies in the following involutions

$$\cdots\ i\ \cdots\ i\pm 1\ \cdots\ i\mp 1\ \cdots \quad \overset{d_i}{\longleftrightarrow} \quad \cdots\ i\mp 1\ \cdots\ i\pm 1\ \cdots\ i\ \cdots$$

$$\cdots\ i\ \cdots\ i\pm 1\ \cdots\ i\mp 1\ \cdots \quad \overset{\widetilde{d}_i}{\longleftrightarrow} \quad \cdots\ i\pm 1\ \cdots\ i\mp 1\ \cdots\ i\ \cdots$$

There is a notion of the *distance* between two entries:

dist$(i,j) < k \quad \Leftrightarrow \quad i,j$ form a potential k-inversion
dist$(i,j) = k \quad \Leftrightarrow \quad i,j$ appear in the same tableau as $\begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}$ or $\begin{array}{|c|}\hline j \\ \hline i \\ \hline \end{array}$

Define an involution $D_i^{(k)}(\mathbf{T}) \ = \begin{cases} d_i(\mathbf{T}) & \text{if dist}(i{-}1,i,i{+}1) > k \\ \widetilde{d}_i(\mathbf{T}) & \text{if dist}(i{-}1,i,i{+}1) \leq k \end{cases}$

Fact: $\mathrm{inv}_k(\mathbf{T}) = \mathrm{inv}_k(D_i^{(k)}(\mathbf{T}))$. Therefore define $\mathbf{T} \overset{i}{\text{———}} D_i^{(k)}(\mathbf{T})$.

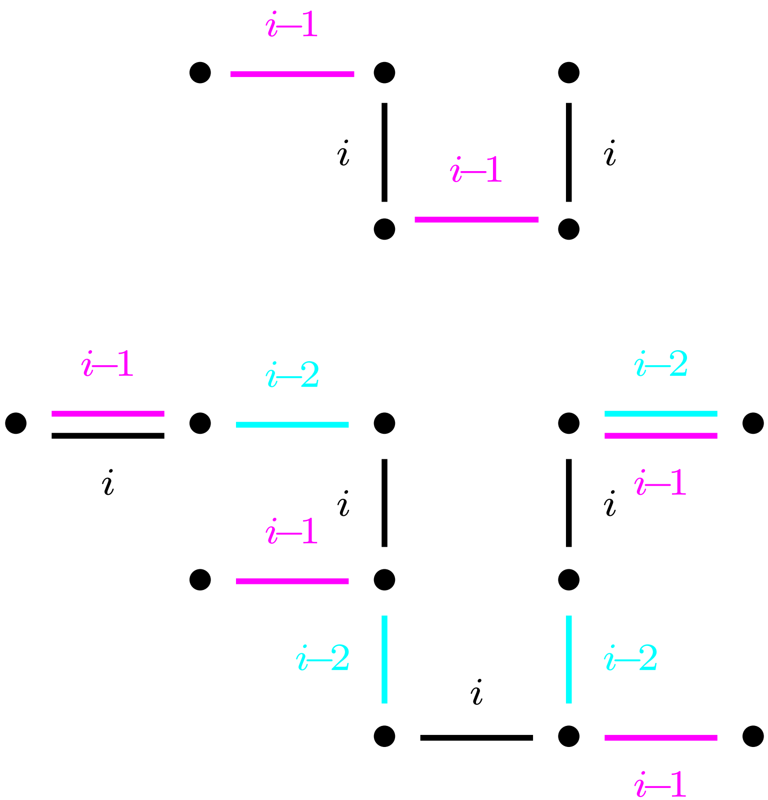
It is straightforward to check that this defines a ~~DEG~~ **D graph**.
The generating function of a **D graph** is Schur positive, and so ...

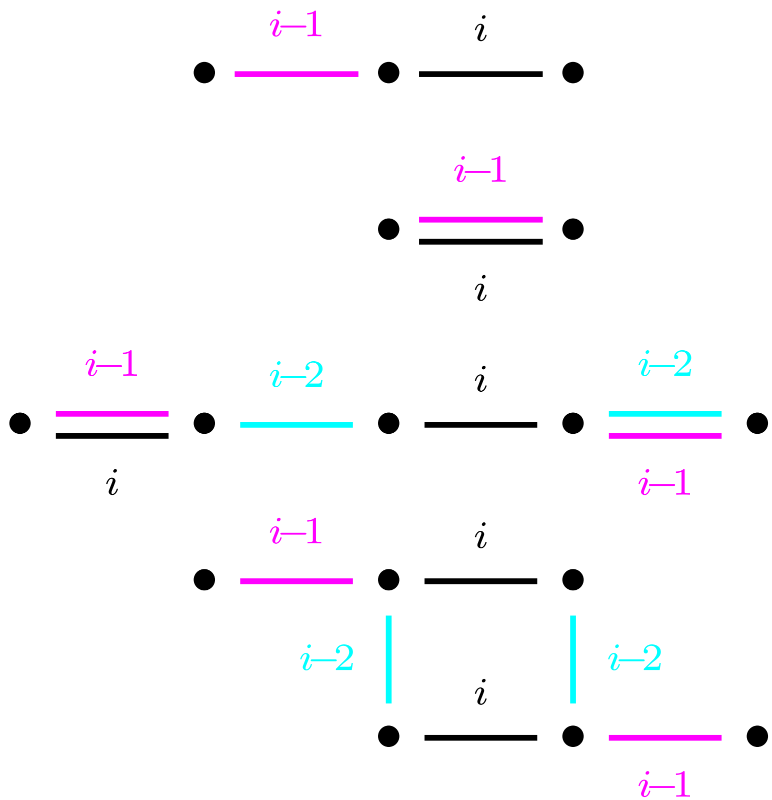
Theorem. (A. 2007) The LLT polynomial $\widetilde{G}_\mu^{(k)}$ is Schur positive.

From D graphs to DEGs

D graphs: vertices of conn comps of $D_{i-2} \cup D_{i-1} \cup D_i$ are in bijection with $\mathrm{SYT}(\lambda^1) \cup \cdots \cup \mathrm{SYT}(\lambda^m)$.

DEGs: vertices of conn comps of $D_{i-2} \cup D_{i-1} \cup D_i$ are in bijection with $\mathrm{SYT}(\lambda)$.

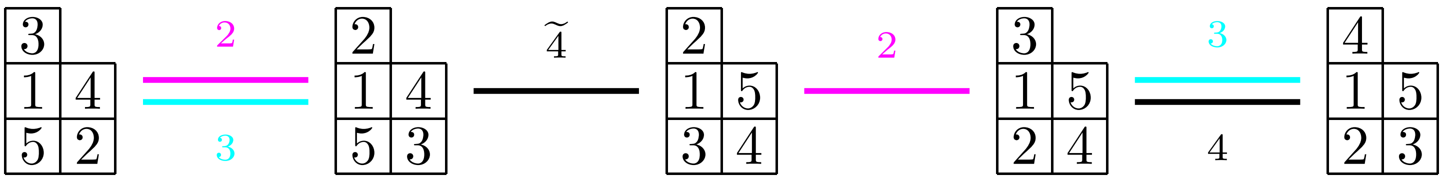




Theorem. (A. 2007) A **D graph** may be transformed into a DEG in such a way that preserves the generating function.

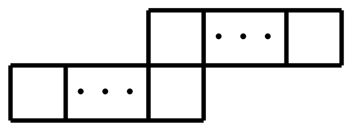
Macdonald polynomials, defined by Macdonald in 1988, may be expressed as the generating function for fillings of a partition diagram (conjectured by Haglund, proved by Haglund-Haiman-Loehr 2005):

$$\widetilde{H}_\mu(\mathbf{x};q,t) = \sum_{S:\mu\rightarrow\mathbb{Z}_+} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} \mathbf{x}^{\mathrm{wt}(S)} = \sum_{S:\mu \xrightarrow{\sim} [n]} q^{\mathrm{inv}(S)} t^{\mathrm{maj}(S)} Q_{\sigma(S)}(\mathbf{x}).$$



Construct a **D graph** on fillings using the following involution:

$$D_i(S) = \begin{cases} \widetilde{d}_i(S) & \text{if } i-1, i, i+1 \text{ fit in} \\ d_i(S) & \text{otherwise.} \end{cases}$$



Theorem. (A. 2007) The Macdonald polynomial $\widetilde{H}_\mu$ is Schur positive.

III. Representation Theory

Weyl groups acting on 0-weight spaces

G = complex, reductive Lie group — W = Weyl group of G

E = finite dim. irrep. of G over $\mathbb{C}$ — $E^0 = 0$-weight space of E

Goal: Determine how the natural action $W \curvearrowright E^0$ decomposes.

Example (studied by Gutkin in 1973):

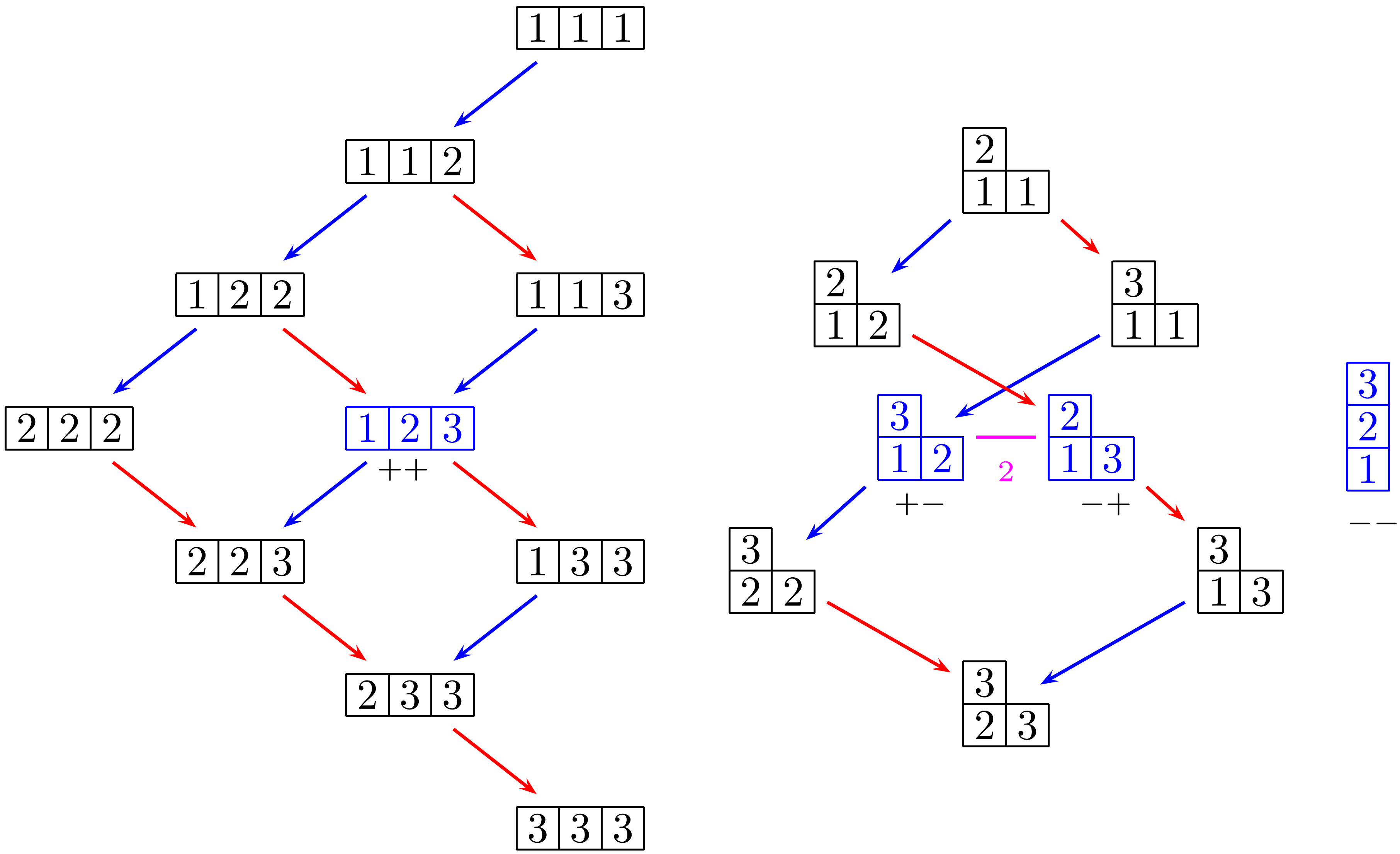
Let $G = \mathrm{SL}_n$, $W \cong \mathcal{S}_n$, and $E =$ an irreducible component of $(\mathbb{C}^n)^{\otimes n}$, then E^0 is an irrep of $\mathcal{S}_n$ and all irreps of $\mathcal{S}_n$ arise in this way.

More generally...

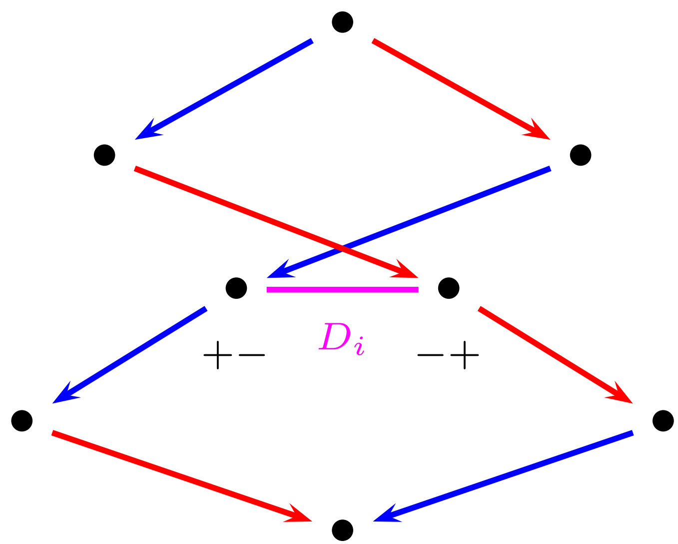
Let E_λ be the irrep of SL_n with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$.
Then E_λ^0 is nontrivial if and only if $\lambda_1 + \cdots + \lambda_n = kn$ for some $k \in \mathbb{Z}_+$.
When this is the case, E_λ^0 consists of all vectors of weight $(k, \cdots, k)$.

Goal: Decompose $\mathcal{S}_n \curvearrowright E_\lambda^0$ into irreps of $\mathcal{S}_n$.

A motivating example



Combinatorially, the 0-weight space of a crystal graph consists of all vertices which lie at the center of each i-string.



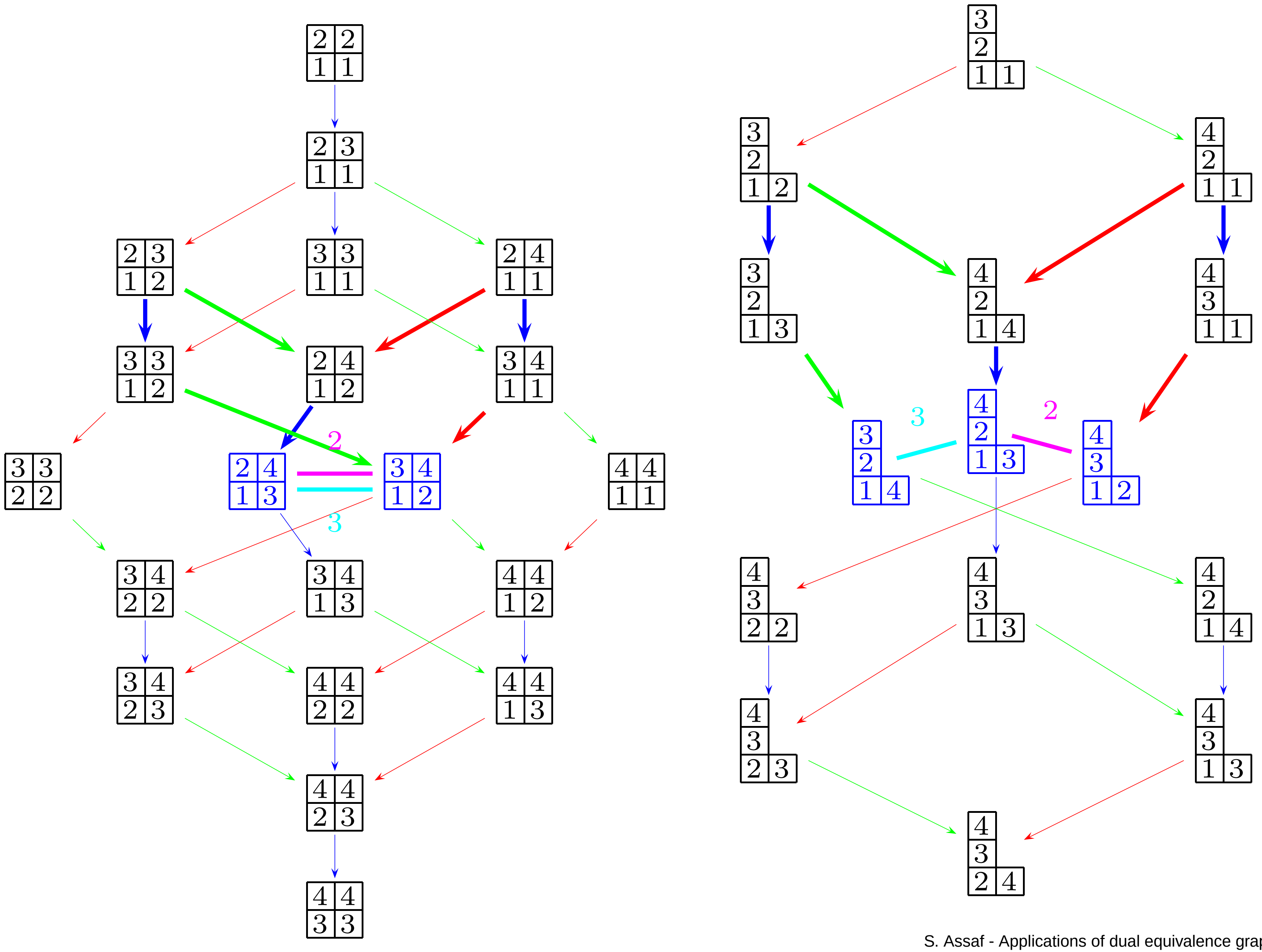
Given a crystal graph $\mathcal{X}$, define a vertex-signed, edge-colored graph $\mathcal{G}(\mathcal{X}) = (V, \sigma, D)$ as follows:

$$V = \{x \in \mathcal{X} \mid \varepsilon(x,i) = -\delta(x,i) = 0 \text{ or } 1 \ \ \forall\, i, \}$$

$$\sigma(x)_i = \begin{cases} + & \text{if } \varepsilon(x,i) = 1 \\ - & \text{if } \varepsilon(x,i) = 0 \end{cases}$$

$$D_i(x) = \begin{cases} \widetilde{f}_{i-1}\widetilde{f}_i\widetilde{e}_{i-1}\widetilde{e}_i x & \text{if } \varepsilon(x,i) = 1 \text{ and } \varepsilon(x,i-1) = 0 \\ \widetilde{f}_i\widetilde{f}_{i-1}\widetilde{e}_i\widetilde{e}_{i-1} x & \text{if } \varepsilon(x,i) = 0 \text{ and } \varepsilon(x,i-1) = 1 \end{cases}$$

Examples of the construction of $\mathcal{G}(\mathcal{X})$

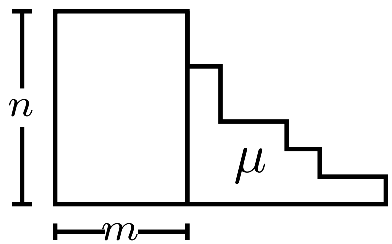


It follows from the local characterization for crystal graphs (Stembridge) that if $\mathcal{X}$ is a crystal graph, the $\mathcal{G}(\mathcal{X})$ is well-defined.

Using the local characterizations for crystal graphs and for dual equivalence graphs, we have the following result.

Theorem. (A. 2007) If $\mathcal{X}$ is a crystal graph with 0-weight space given by V, then $\mathcal{G}(\mathcal{X})$ is a dual equivalence graph.

Corollary. Let $\lambda = (\mu_1 + m, \mu_2 + m, \ldots, \mu_n + m)$ for some partition μ of n and some integer $m > 0$. Then $\mathcal{G}(\mathcal{X}_\lambda^n) = \mathcal{G}_\mu$.



For GL_n, we need to modify the signatures based on the parity of m:

$$\sigma(x)_i = \begin{cases} + & \text{if } \varepsilon(x,i) + m \text{ is even} \\ - & \text{if } \varepsilon(x,i) + m \text{ is odd} \end{cases} \rightsquigarrow \mathcal{G}(\mathcal{X}_\lambda^n) = \begin{cases} \mathcal{G}_\mu & \text{for } m \text{ even} \\ \mathcal{G}_{\mu'} & \text{for } m \text{ odd} \end{cases}$$

Extensions

In general, V consists of all vertices of $\mathcal{X}$ at the center of each i-string.

Let E be the irrep of SL_4 with highest weight $\lambda = (4,3,1,0)$.

$$V = \left\{ \begin{array}{llll} 4 & 3\,3\,4 \text{ over } & & \end{array} \right.$$

$$V = \left\{ \begin{array}{cccc} \begin{array}{|c|c|c|c|} \hline 4 \\ \hline 2 & 3 & 3 \\ \hline 1 & 1 & 2 & 4 \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 4 \\ \hline 3 & 3 & 4 \\ \hline 1 & 1 & 2 & 2 \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 4 \\ \hline 2 & 2 & 4 \\ \hline 1 & 1 & 3 & 3 \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 3 \\ \hline 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 4 \\ \hline \end{array} \\ & \begin{array}{|c|c|c|c|} \hline 4 \\ \hline 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 3 \\ \hline 2 & 4 & 4 \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} & \begin{array}{|c|c|c|c|} \hline 3 \\ \hline 2 & 2 & 4 \\ \hline 1 & 1 & 3 & 4 \\ \hline \end{array} \end{array} \right\}$$

The decomposition into irreps of $\mathcal{S}_4$ is $E^0 \cong S^{1,1,1,1} \oplus S^{2,1,1,} \oplus S^{3,1}$.

The edges of $\mathcal{G}(\mathcal{X})$ are not determined solely by $\widetilde{e}_{i-1}, \widetilde{f}_{i-1}, \widetilde{e}_i, \widetilde{f}_i$.
The edges cannot be defined by restriction and *jeu de taquin*.

A rule for edges of $\mathcal{G}(\mathcal{X})$ has been worked out in several cases (sometimes giving a **D graph**), but the general rule is still a mystery.