Blocks and Counting Conjectures

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1. Simple modules

- $\mathbb{F}$ algebraically closed field, $\text{char}(F) = p > 0$
- G finite group
- $\mathbb{F}G = \{\sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{F} \text{ for } g \in G\}$ group algebra

Modular representation theory is concerned with

$$\mathbb{F}G - \textbf{mod},$$

the category of finitely generated (left) $\mathbb{F}G$-modules. (Some related categories are also important.) One way to try to understand a finitely generated module is to see how it is built up from "simple" modules.

Definition

For a finite-dimensional $\mathbb{F}$-algebra A (e.g. $A = \mathbb{F}G$), an A-module $L \neq 0$ is called **simple** if 0 and L are the only submodules of L.

Then L is certainly finitely generated; in fact, $L = Ax$ whenever $0 \neq x \in L$. In the following, we denote the number of isomorphism classes of simple A-modules by

$$\ell(A).$$

Then $\ell(A) \leq \dim A < \infty$.

Question

What is $\ell(\mathbb{F}G)$?

R. Brauer 1935

$\ell(\mathbb{F}G)$ equals the number of p-regular conjugacy classes of G.

Here a conjugacy class K of G is called ***p*-regular** (or a **p'-conjugacy class**) if the order of the elements in K is not divisible by p. In the following, we denote the set of *all* conjugacy classes of G by

$$\mathrm{Cl}(G).$$

For a finite-dimensional $\mathbb{F}$-algebra A, we denote the **center** of A by

$$Z(A) := \{z \in A : za = az \text{ for } a \in A\},$$

and we set

$$\mathrm{k}(A) := \dim Z(A).$$

Then $Z(A)$ is a commutative subalgebra of A, and

$$Z(\mathbb{F}G) = \{ \sum_{K \in \mathrm{Cl}(G)} \alpha_K K^+ : \alpha_K \in \mathbb{F} \text{ for } K \in \mathrm{Cl}(G)\}.$$

Here, and in the following, we set

$$X^+ := \sum_{x \in X} x \in \mathbb{F}G,$$

for $X \subseteq G$. Thus the **class sums** K^+, with $K \in \mathrm{Cl}(G)$, form an $\mathbb{F}$-basis of $Z(\mathbb{F}G)$. Hence

$$\begin{aligned} \ell(\mathbb{F}G) &= |\{K \in \mathrm{Cl}(G) : K \ p\text{-regular}\}| \\ &\leq |\mathrm{Cl}(G)| = \dim Z(\mathbb{F}G) = \mathrm{k}(\mathbb{F}G) =: \mathrm{k}(G). \end{aligned}$$

Numbers like $\mathrm{k}(\mathbb{F}G)$ and $\ell(\mathbb{F}G)$ play a major role in counting conjectures.

Example. If $p \not| \, |G|$ then every $K \in \mathrm{Cl}(G)$ is p-regular, so in this case $\ell(\mathbb{F}G) = \mathrm{k}(\mathbb{F}G)$.

2. Composition series

- A finite-dimensional $\mathbb{F}$-algebra (e.g. $A = \mathbb{F}G$ for a finite group G)
- $M \in A-\mathbf{mod}$

Definition

A **composition series** of M is a sequence of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M$$

such that $M_1/M_0, \ldots, M_r/M_{r-1}$ are simple A-modules.

The following uniqueness result is essential.

Theorem (Jordan-Hölder)

Let

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_r = M \text{ and } 0 = N_0 \subseteq N_1 \subseteq \ldots \subseteq N_s = M$$

be composition series of M. Then $r = s$, and there is a permutation π of $\{1, \ldots, r\}$ such that

$$M_i/M_{i-1} \simeq_A N_{\pi(i)}/N_{\pi(i)-1} \text{ for } i = 1, \ldots, r.$$

Thus M has a well-defined **composition length** r, and **composition factors** $M_1/M_0, \ldots, M_r/M_{r-1}$ which are unique up to order and isomorphism. For a simple A-module L, we denote by

$$[M : L]$$

the (**Jordan-Hölder**) **multiplicity** of L in M, i.e. the number of composition factors of M which are isomorphic to L.

3. Indecomposable modules

- A finite-dimensional $\mathbb{F}$-algebra
- $M \in A-\mathbf{mod}$

Definition

M is called **indecomposable** if $M \neq 0$ and if M does not have a decomposition $M = M' \oplus M''$ where M', M'' are proper submodules of M.

Every simple module is indecomposable; the converse is false, in general. The following uniqueness result is very important.

Theorem (Krull-Schmidt)

Let $M = M_1 \oplus \cdots \oplus M_r = N_1 \oplus \cdots \oplus N_s$ where $M_1, \ldots, M_r, N_1, \ldots, N_s$ are indecomposable submodules of M. Then $r = s$, and there is a permutation π of $\{1, \ldots, r\}$ such that $M_i \simeq_A N_{\pi(i)}$ for $i = 1, \ldots, r$.

Then $M_1, \ldots, M_r$ are called **components** of M. By the Krull-Schmidt theorem, they are *essentially* unique. For $N \in A-\mathbf{mod}$, we write

$$N \mid M$$

if N is isomorphic to a direct summand of M. Whereas the number of isomorphism classes of *simple* A-modules is always finite, the number of isomorphism classes of *indecomposable* A-modules is usually infinite.

Definition

A is said to have **finite representation type** if A has only finitely many indecomposable modules, up to isomorphism. Otherwise, A is said to have **infinite representation type**.

Using the theory of vertices and sources, one can show that the group algebra $\mathbb{F}G$ of a finite group G over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$ has finite representation type iff the Sylow p-subgroups of G are cyclic. For example, if $G = \langle g \rangle$ is a cyclic p-group of order $q = p^n$ then $\mathbb{F}G$ has precisely q indecomposable modules $M_1, \ldots, M_q$, up to isomorphism. Here each M_j has dimension j and an $\mathbb{F}$-basis such that the action of g on M_j is given by a Jordan block of size j:

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix}.$$

4. Radical and socle

- A finite-dimensional $\mathbb{F}$-algebra
- $M \in A-\mathbf{mod}$

Definition

The **socle** $\mathrm{Soc}(M)$ is the sum of all simple submodules of M. If $\mathrm{Soc}(M) = M$ then M is called **semisimple** (or **completely reducible**).

Thus $\mathrm{Soc}(M)$ is the largest semisimple submodule of M, and one can write

$$\mathrm{Soc}(M) = L_1 \oplus \cdots \oplus L_t$$

with simple submodules $L_1, \ldots, L_t$ of M.

Definition

The **radical** $\mathrm{Rad}(M)$ is the intersection of all maximal submodules of M.

Then $M/\mathrm{Rad}(M)$ is the largest semisimple factor module of M.

Every finite-dimensional $\mathbb{F}$-algebra A can also be considered as a left A-module; then we write ${}_AA$ and call this the **regular** (left) A-module. Furthermore, $J(A) := \mathrm{Rad}({}_AA)$ is called the (**Jacobson**) **radical** of A. Then $J(A)$ is always a (two-sided) ideal of A; in fact, it is the largest nilpotent ideal of A, in the following sense:

Definition

An ideal I of A is called **nilpotent** if $I^n = 0$ for some $n \in \mathbb{N}$. Here I^n is spanned by all elements of the form $x_1 \ldots x_n$ where $x_1, \ldots, x_n \in I$.

The following structure theorem is very important:

Theorem (Wedderburn)

There is an isomorphism of $\mathbb{F}$-algebras

$$(\mathcal{W}) \qquad A/J(A) \cong \mathbb{F}^{d_1 \times d_1} \times \cdots \times \mathbb{F}^{d_l \times d_l},$$

for suitable $d_1, \ldots, d_l \in \mathbb{N}$. Moreover, $d_1, \ldots, d_l$ are uniquely determined up to their order.

Here $\mathbb{F}^{d \times d}$ denotes the $\mathbb{F}$-algebra of all $d \times d$-matrices with entries in $\mathbb{F}$, for $d \in \mathbb{N}$. It is easy to see that $\mathbb{F}^{d \times d}$ has a unique simple module, up to isomorphism; it is given by $\mathbb{F}^d$, the set of all column vectors, with $\mathbb{F}^{d \times d}$ acting by left multiplication.

Thus, by inflation, every factor $\mathbb{F}^{d_i \times d_i}$ on the right hand side of $(\mathcal{W})$ gives rise to a simple A-module L_i of dimension d_i. In fact, every simple A-module arises in this way (up to isomorphism) since

$$J(A) = \{a \in A : aL = 0 \text{ for every simple } A\text{-module } L\}.$$

The radical of A is related to the radical of an A-module M via

$$\mathrm{Rad}(M) = J(A)M.$$

For the group algebra $\mathbb{F}G$, we have the following important result:

Theorem (MASCHKE)

$J(\mathbb{F}G) = 0 \iff p \nmid |G| \iff$ every $\mathbb{F}G$-module is semisimple.

For $p = \mathrm{char}(\mathbb{F}) \mid |G|$, it is not easy to compute $J(\mathbb{F}G)$.

6. Projective modules

Let A be a finite-dimensional $\mathbb{F}$-algebra.

Definition

(i) An A-module F is called **free** if $F \simeq_A A^n$ for some $n \in \mathbb{N}_0$.

(ii) An A-module P is called **projective** if $P \mid F$ for a free A-module F.

There are a number of other important characterizations of projective modules (e.g. by a universal property) which we omit. Since every projective module decomposes into indecomposable projective modules, these are especially important.

Proposition

Every indecomposable projective A-module P has a unique maximal submodule. Thus $P/\mathrm{Rad}(P)$ is a simple A-module. Moreover, the map

$$P \longmapsto P/\mathrm{Rad}(P)$$

induces a bijection

$$\{\text{isomorphism classes of indecomposable projective } A\text{-modules}\}$$
$$\updownarrow$$
$$\{\text{isomorphism classes of simple } A\text{-modules}\}.$$

For a group algebra $\mathbb{F}G$, the following holds.

Proposition

If $P \in \mathbb{F}G-\mathbf{mod}$ is indecomposable and projective then

$$\mathrm{Soc}(P) \simeq P/\mathrm{Rad}(P).$$

7. The Cartan matrix

- A finite-dimensional $\mathbb{F}$-algebra
- $P_1, \ldots, P_l$ representatives for the isomorphism classes of indecomposable projective A-modules
- $L_i = P_i/\mathrm{Rad}(P_i)$ $(i = 1, \ldots, l)$

Then $L_1, \ldots, L_l$ represent the isomorphism classes of simple A-modules.

Definition

The **Cartan invariants** of A are defined by

$$c_{ij} := [P_i : L_j] \in \mathbb{N}_0 \qquad (i, j = 1, \ldots, l).$$

Note that $c_{ii} \geq 1$ for $i = 1, \ldots, l$. Then $C := (c_{ij}) \in \mathbb{Z}^{l \times l}$ is called the **Cartan matrix** of A. (It is only unique up to a renumbering of $P_1, \ldots, P_l$.)

For a group algebra $\mathbb{F}G$, the Cartan matrix C is symmetric (i.e. $c_{ij} = c_{ji}$ for $i, j = 1, \ldots, l$) and positive definite (i.e. $x^\top Cx > 0$ whenever $0 \neq x \in \mathbb{R}^l$). R. Brauer proved that $\det(C)$ is a power of p; in fact,

$$\det(C) = |S_1| \cdots |S_l|$$

where $x_1, \ldots, x_l$ form a transversal for the p-regular conjugacy classes of G and $S_i \in \mathrm{Syl}_p(C_G(x_i))$ for $i = 1, \ldots, l$. More precisely, $|S_1|, \ldots, |S_l|$ are the elementary divisors of the integer matrix C.

The Cartan invariants are important numbers which can be attached to (group) algebras.

8. Idempotents

Let A be a finite-dimensional $\mathbb{F}$-algebra.

Definition

(i) An **idempotent** in A is an element $e \in A$ such that $e^2 = e$.

(ii) Two idempotents $e, f \in A$ are called **orthogonal** if $ef = 0 = fe$.

If e is an idempotent in A then $1 - e$ is also an idempotent in A, and $e, 1 - e$ are orthogonal. This implies that

$$A = Ae \oplus A(1 - e).$$

Thus Ae is a projective A-module.

Definition

An idempotent $e \in A$ is called **primitive** (in A) if it is impossible to write $e = e' + e''$ with nonzero orthogonal idempotents $e', e'' \in A$.

It is easy to see that an idempotent $e \in A$ is primitive in A iff the projective A-module Ae is indecomposable. Moreover, for idempotents $e, f \in A$, we have $Ae \simeq_A Af$ iff e and f are **conjugate** in A, i.e. $f = ueu^{-1}$ for an invertible element (**unit**) u in A. In this way, the map

$$e \longmapsto Ae$$

induces a bijection

$$\{\text{conjugacy classes of primitive idempotents in } A\}$$

$$\updownarrow$$

$$\{\text{isomorphism classes of indecomposable projective } A\text{-modules}\}.$$

9. Blocks

Let A be a finite-dimensional $\mathbb{F}$-algebra.

Definition

A **block idempotent** of A is an idempotent $e \in Z(A)$ which is primitive in $Z(A)$.

Then A contains only finitely many block idempotents $e_1, \dots, e_r$. They are pairwise orthogonal, and

$$(*) \qquad 1 = e_1 + \cdots + e_r.$$

For $i = 1, \dots, r$, the ideal $B_i = Ae_i = e_iA$ of A is called a **block** (**ideal**) of A. Each block B_i is an $\mathbb{F}$-algebra in its own right, with identity element e_i. Thus B_i is also called a **block algebra**. Furthermore, $(*)$ implies that A has a **block decomposition**

$$(**) \qquad A = B_1 \oplus \cdots \oplus B_r.$$

If $M \in A-\mathbf{mod}$ then

$$M = B_1 M \oplus \cdots \oplus B_r M$$

with submodules $B_1 M, \ldots, B_r M$. In particular, if M is indecomposable then $B_i M \neq 0$ for a unique $i \in \{1, \ldots, r\}$. Thus $M = B_i M$ can be viewed as a B_i-module, and $B_j M = 0$ for $j \neq i$. We say that M **belongs to** the block B_i.

In this way we get a partition of the indecomposable A-modules according to the blocks $B_1, \ldots, B_r$ of A:

$$\begin{aligned}&\{\text{isomorphism classes of indecomposable } A\text{-modules}\}\\ &= \bigsqcup_{i=1}^{r} \{\text{isomorphism classes of indecomposable } B_i\text{-modules}\}.\end{aligned}$$

In particular, we get a partition of the simple A-modules according to the blocks $B_1, \ldots, B_r$ of A:

$$\begin{aligned}&\{\text{isomorphism classes of simple } A\text{-modules}\}\\ &= \bigsqcup_{i=1}^{r} \{\text{isomorphism classes of simple } B_i\text{-modules}\}.\end{aligned}$$

In particular, we have

$$\ell(A) = \ell(B_1) + \cdots + \ell(B_r).$$

In a similar way, we also have

$$\mathrm{k}(A) = \mathrm{k}(B_1) + \cdots + \mathrm{k}(B_r).$$

The block decomposition also induces a partition of the Cartan matrix C of A:

$$C = \begin{pmatrix} C_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & C_r \end{pmatrix}$$

where each C_i is the Cartan matrix of the block B_i.

In block theory, we try to understand the structure of a fixed block $B = B_i$ of a group algebra $\mathbb{F}G$.

10. Defect groups

- G finite group
- B block of $\mathbb{F}G$ with block idempotent e

Since $e \in Z(\mathbb{F}G)$, we can write e as a linear combination of class sums:

$$e = \sum_{K \in \mathrm{Cl}(G)} \alpha_K K^+ \qquad (\alpha_K \in \mathbb{F} \text{ for } K \in \mathrm{Cl}(G)).$$

For $K \in \mathrm{Cl}(G)$, the elements in

$$\mathrm{Def}(K) := \mathrm{Def}_p(K) := \bigcup_{g \in K} \mathrm{Syl}_p(C_G(g))$$

are called the (p)-**defect groups** of K. Thus $\mathrm{Def}_p(K)$ is a conjugacy class of p-subgroups of G.

Brauer has shown that there exists a p-subgroup D of G with the following properties:

(i) $D \in \mathrm{Def}(K)$ for some $K \in \mathrm{Cl}(G)$ with $\alpha_K \neq 0$;

(ii) Whenever $S \in \mathrm{Def}(L)$ for some $L \in \mathrm{Cl}(G)$ with $\alpha_L \neq 0$ then S is conjugate to a subgroup of D. (We write $S \leq_G D$.)

Then D is called a **defect group** of B. Thus the defect groups of B form a conjugacy class of p-subgroups of G which we denote by

$$\mathrm{Def}(B).$$

If $|D| = p^d$ then d is called the **defect** of B. There are many results (and many open conjectures) which relate the structure of a block to the structure of its defect groups. As an example, we mention:

BRAUER's $\mathrm{k}(B)$-conjecture

Let B be a block of the group algebra $\mathbb{F}G$ with defect group D. Then $\mathrm{k}(B) = \dim Z(B) \leq |D|$.

This conjecture has in recent years been proved by the combined efforts of several people for p-solvable groups, but remains open in general. Another open problem is the following one:

DONOVAN's conjecture

For any finite p-group D, there are only finitely many Morita equivalence classes of blocks of group algebras which have a defect group which is isomorphic to D.

Donovan's conjecture has been proved for several classes of finite groups, such as p-solvable groups and symmetric groups, but also remains open in general.

11. Morita equivalence

Definition

Two finite-dimensional $\mathbb{F}$-algebras A, B are called **Morita equivalent** if their module categories $A-\mathbf{mod}$ and $B-\mathbf{mod}$ are equivalent (as $\mathbb{F}$-linear categories). This means that there are $\mathbb{F}$-linear functors

$$\Phi : A-\mathbf{mod} \longrightarrow B-\mathbf{mod} \text{ and } \Psi : B-\mathbf{mod} \longrightarrow A-\mathbf{mod}$$

such that $\Psi \circ \Phi \sim \mathrm{Id}_{A-\mathbf{mod}}$ and $\Phi \circ \Psi \sim \mathrm{Id}_{B-\mathbf{mod}}$; here $\mathrm{Id}_{A-\mathbf{mod}}$ and $\mathrm{Id}_{B-\mathbf{mod}}$ denote the identity functors on $A-\mathbf{mod}$ and $B-\mathbf{mod}$, respectively.

This means that A and B – while perhaps not isomorphic – have the "same" representation theory. In modern representation theory, equivalences between module categories (and related categories) are extremely important tools.

In general, functors $\Gamma, \Delta : \mathcal{C} \longrightarrow \mathcal{D}$ between categories $\mathcal{C}, \mathcal{D}$ are called **naturally equivalent** if there exists a family $\phi = (\phi_X)_{X \in \mathcal{C}}$ of morphisms $\phi_X : \Gamma X \longrightarrow \Delta X$ in $\mathcal{D}$ such that

$$(\Delta f) \circ \phi_X = \phi_Y \circ (\Gamma f)$$

for every morphism $f : X \longrightarrow Y$ in $\mathcal{C}$:

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\phi_X} & \Delta X \\ \downarrow \Gamma f & & \downarrow \Delta f \\ \Gamma Y & \xrightarrow{\phi_Y} & \Delta Y \end{array}$$

Then ϕ is called a **natural equivalence** between Γ and Δ.

Morita has shown that Morita equivalences between $\mathbb{F}$-algebras can also be characterized in the following way:

Theorem (Morita)

For finite-dimensional $\mathbb{F}$-algebras A, B, the following assertions are equivalent:

(1) A, B are Morita equivalent;

(2) There exist an A-B-bimodule M and a B-A-bimodule N such that

$$M \otimes_B N \simeq A \quad \text{(as } A\text{-}A\text{-bimodules)}$$

and

$$N \otimes_A M \simeq B \quad \text{(as } B\text{-}B\text{-bimodules)};$$

(3) B° is isomorphic to $\mathrm{End}_A(P)$, for some A-progenerator P.

Here B° denotes the **opposite algebra** of B, and an A-**progenerator** is a finitely generated projective A-module P which is also a **generator** (i.e. $A = \sum_{\phi \in \mathrm{Hom}_A(P,A)} \phi(P)$).

Examples. (i) Isomorphic algebras are always Morita equivalent.

(ii) For $n \in \mathbb{N}$, A is always Morita equivalent to $A^{n \times n}$.

(iii) If e is an idempotent in A such that $AeA = A$ then A is Morita equivalent to eAe.

Remarks. (i) Morita equivalent algebras have the same number of simple modules, the same representation type, the same Cartan matrices and isomorphic centers.

(ii) Two commutative algebras are Morita equivalent iff they are isomorphic.

(iii) Two local algebras are Morita equivalent iff they are isomorphic.

We recall that a finite-dimensional $\mathbb{F}$-algebra A is called **local** if $A/J(A) \cong \mathbb{F}$.

12. Blocks and subgroups

Let A be a finite-dimensional $\mathbb{F}$-algebra.

Remarks. (i) An A-module M is indecomposable iff $E := \mathrm{End}_A(M)$ is a local $\mathbb{F}$-algebra (i.e. $E/J(E) \cong \mathbb{F}$).

(ii) An idempotent $e \in A$ is primitive in A iff the $\mathbb{F}$-algebra eAe is local.

(iii) For every block B of A, the center $Z(B)$ is a local $\mathbb{F}$-algebra. This yields a homomorphism of $\mathbb{F}$-algebras

$$\omega_B : Z(A) \longrightarrow Z(B) \longrightarrow Z(B)/J(Z(B)) \cong \mathbb{F}.$$

Then ω_B is called the **central character** of B.

(iv) The map $B \longmapsto \omega_B$ gives a bijection

$$\{\text{blocks of } A\}$$

$$\updownarrow$$

$$\{\text{homomorphisms of } \mathbb{F}\text{-algebras } Z(A) \longrightarrow \mathbb{F}\}.$$

Definition

Let H be a subgroup of a finite group G, and let b be a block of $\mathbb{F}H$. Let

$$\mathrm{Pr}_H^G : \mathbb{F}G \longrightarrow \mathbb{F}H, \quad \sum_{g \in G} \alpha_g g \longmapsto \sum_{g \in H} \alpha_g g,$$

denote the natural projection. If $\omega_b \circ \mathrm{Pr}_H^G | Z(\mathbb{F}G)$ is a homomorphism of $\mathbb{F}$-algebras then $\omega_b \circ \mathrm{Pr}_H^G | Z(\mathbb{F}G) = \omega_B$ for a unique block B of $\mathbb{F}G$. We call $B = b^G$ the **induced block**.

Remarks. (i) There are different ways to define induced blocks.

(ii) In general, there is no guarantee that $\omega_b \circ \mathrm{Pr}_H^G | Z(\mathbb{F}G)$ is a homomorphism of $\mathbb{F}$-algebras.

(iii) If $C_G(D) \subseteq H$ for a defect group D of b then b^G is defined; in this case, b is called **admissible** in G.

(iv) If b^G is defined then there is a defect group E of b^G such that $D \subseteq E$.

13. The First Main Theorem on Blocks

- $\mathbb{F}$ algebraically closed field, $\text{char}(\mathbb{F}) = p > 0$
- G finite group

Brauer's First Main Theorem on Blocks

Let D be a p-subgroup of G. Then the map $b \longmapsto b^G$ gives a bijection

$$\{\text{blocks of } \mathbb{F}N_G(D) \text{ with defect group } D\}$$

$$\updownarrow$$

$$\{\text{blocks of } \mathbb{F}G \text{ with defect group } D\}.$$

Definition

In the situation above, b^G is called the **Brauer correspondent** of b in G, and b is called the **Brauer correspondent** of b^G in $N_G(D)$.

Remark. (i) Green has shown that, for every defect group D of a block B of $\mathbb{F}G$ and for every Sylow p-subgroup S of G containing D, there exists an element $g \in C_G(D)$ such that

$$D = S \cap gSg^{-1};$$

in particular, $O_p(G) \subseteq D$. (This last fact is an earlier result of Brauer.) This implies that $D = O_p(N_G(D))$. Subgroups Q of G with $Q = O_p(N_G(Q))$ are called **radical** p-subgroups of G.

(ii) For any p-subgroup D of G, the number of blocks of $\mathbb{F}G$ with defect group D is bounded above by the number of p-regular conjugacy classes of G with defect group D.

(iii) Robinson has expressed the number of blocks with defect group D as the rank of a certain matrix.

(iv) For $D \in \mathrm{Syl}_p(G)$, the number of blocks with defect group D equals the number of p-regular conjugacy classes with defect group D (Brauer).

Brauer's First Main Theorem is a good example of a result relating the representations of a finite group G to the representations of its "local" subgroups. Here a **local** subgroup of G is a subgroup of the form $N_G(Q)$ where Q is a nontrivial p-subgroup of G.

14. Blocks and normal subgroups

Remark. Let G be a finite group and $K \trianglelefteq G$.

(i) Then G acts by conjugation on $\mathbb{F}K$, permuting the blocks of $\mathbb{F}K$.

(ii) Let B be a block of $\mathbb{F}G$, and let b be a block of $\mathbb{F}K$. If $Bb \neq 0$ then B is said to **cover** b.

Proposition

For every block B of $\mathbb{F}G$, the blocks of $\mathbb{F}K$ covered by B form a G-orbit. Thus their number is $|G : I|$ where b is a block of $\mathbb{F}K$ covered by B, and

$$I := I_G(b) := \{g \in G : gbg^{-1} = b\}$$

denotes the **inertia subgroup** of b.

Proposition (Fong-Reynolds)

Let b be a block of $\mathbb{F}K$ and $I := I_G(b)$.

(i) Then the map $\beta \longmapsto \beta^G$ is a bijection

$$\begin{array}{c}\{\text{blocks of } \mathbb{F}I \text{ covering } b\} \\ \updownarrow \\ \{\text{blocks of } \mathbb{F}G \text{ covering } b\}.\end{array}$$

(ii) If $D \in \mathrm{Def}(\beta)$ then $D \in \mathrm{Def}(\beta^G)$ and $D \cap K \in \mathrm{Def}(b)$.

(iii) There is always at least one block β_0 of $\mathbb{F}I$ covering b such that $p \nmid |I : D_0K|$ for $D_0 \in \mathrm{Def}(\beta_0)$.

$$\begin{array}{c} G \\ | \\ I = I_G(b) \\ | \\ K, b \end{array}$$

One can also show that, in the situation above, the blocks β and β^G are always Morita equivalent.

These facts lead to an extension of BRAUER's First Main Theorem.

Proposition

Let D be a p-subgroup of G. Then the map $b \longmapsto b^G$ gives a bijection between the set of $N_G(D)$-orbits of blocks b of $\mathbb{F}DC_G(D)$ with defect group D and $p \nmid |I_{N_G(D)}(b) : DC_G(D)|$ and the set of blocks of $\mathbb{F}N_G(D)$ with defect group D.

15. Blocks and factor groups

- G finite group, $N \trianglelefteq G$
- $\nu : \mathbb{F}G \longrightarrow \mathbb{F}[G/N]$ the canonical map

Remark. If B is a block of $\mathbb{F}G$ then

$$\nu(B) = \overline{B}_1 \oplus \cdots \oplus \overline{B}_t$$

with blocks $\overline{B}_1, \ldots, \overline{B}_t$ of $\mathbb{F}[G/N]$ (possibly $t = 0$). We say that $\overline{B}_1, \ldots, \overline{B}_t$ are **dominated** by B. If $D \in \mathrm{Def}(B)$ then $\overline{B}_1, \ldots, \overline{B}_t$ have defect groups contained in DN/N, and at least one $\overline{B}_i$ has defect group DN/N.

Proposition

Let Q be a p-subgroup of G such that $G = QC_G(Q)$. Then block domination gives rise to a bijection

$$\{\text{blocks of } \mathbb{F}G\}$$
$$\updownarrow$$
$$\{\text{blocks of } \mathbb{F}[G/Q]\}$$

denoted by $B \longmapsto \overline{B}$. If $D \in \mathrm{Def}(B)$ then $[Q \subseteq D$ and$]$ $D/Q \in \mathrm{Def}(\overline{B})$. The Cartan matrices $C, \overline{C}$ of $B, \overline{B}$ are related by

$$C = |Q|\overline{C}.$$

A combination of several results above leads to:

Brauer's Extended First Main Theorem on Blocks

Let D be a p-subgroup of G. Then the results above lead to a bijection

$$\{\text{blocks of } \mathbb{F}G \text{ with defect group } D\}$$
$$\updownarrow$$
$$\{N_G(D)/D\text{-orbits of blocks } \flat \text{ of defect } 0$$
$$\text{in } \mathbb{F}[DC_G(D)/D] \text{ with } p \nmid |I_{N_G(D)/D}(\flat) : DC_G(D)/D|\}.$$

Remark. This means that counting blocks with a fixed defect group is essentially equivalent to counting blocks of defect 0 in a related (smaller) group.

16. The principal block

Recall that every indecomposable $\mathbb{F}G$-module belongs to a unique block of $\mathbb{F}G$. The **trivial** $\mathbb{F}G$-module is the field $\mathbb{F}$ on which G acts via

$$g\alpha := \alpha \quad \text{for} \quad g \in G, \; \alpha \in \mathbb{F}.$$

It is certainly simple and therefore indecomposable. Thus it belongs to a unique block B_0 of $\mathbb{F}G$, the **principal block** of $\mathbb{F}G$. It is easy to see that the defect groups of B_0 are just the Sylow p-subgroups of G.

BRAUER's Third Main Theorem on Blocks

Let H be a subgroup of G, and let b be an admissible block of $\mathbb{F}H$ [so that b^G is defined]. Then b is the principal block of $\mathbb{F}H$ iff b^G is the principal block of $\mathbb{F}G$.

Remark. The principal block is the most important block for all applications of modular representation theory to finite group theory. One of these applications is the following result:

GLAUBERMAN's Z^*-Theorem

Let $S \in \mathrm{Syl}_2(G)$, and let $u \in S$ be an involution (i.e. $u^2 = 1 \neq u$). Suppose that $gug^{-1} = u$ for every $g \in G$ such that $gug^{-1} \in S$. Then $G = C_G(u)O_{2'}(G)$; in particular, G is not simple.

Here $O_{p'}(G)$ denotes the largest normal subgroup of G whose order is not divisible by p.

It has been verified, making use of the classification of finite simple groups, that Glauberman's Z^*-Theorem has an analog for odd primes; however, a direct proof is still missing and remains a challenge for representation theory.

The experts also believe that there should be an analog of Glauberman's Z^*-Theorem for blocks, giving a Morita equivalence between a block B of $\mathbb{F}G$ with defect group D and a block b of $\mathbb{F}C_G(u)$ with $b^G = B$, for suitable $u \in D$. A precise statement needs the concept of a Brauer pair which we will get to soon.

In this way, results in local group theory are often a guideline for results in representation theory.

17. Blocks of defect zero

Theorem (Brauer)

For a block B of a group algebra $\mathbb{F}G$, the following assertions are equivalent:

(1) B has defect zero;

(2) $B \cong \mathbb{F}^{n \times n}$ for some $n \in \mathbb{N}$;

(3) $\mathrm{k}(B) = 1$;

(4) There exists a simple projective $\mathbb{F}G$-module belonging to B.

Remark. (i) This result can be seen as a generalization of Maschke's Theorem.

(ii) The result implies that, for every block B of $\mathbb{F}G$ of defect zero, there is a unique (up to isomorphism) simple $\mathbb{F}G$-module L belonging to B, and L is also projective.

(iii) Thus there is a natural bijection

$$\{\text{blocks of defect 0 of } \mathbb{F}G\} \\ \updownarrow \\ \{\text{isomorphism classes of simple projective } \mathbb{F}G\text{-modules}\}$$

(iv) The result says that the structure of blocks of defect zero is easy. In the following, we will present a number of further structure theorems.

18. Brauer pairs

Brauer pairs (also called subpairs) were introduced by ALPERIN and BROUÉ, extending an earlier concept of BRAUER.

Definition

A **Brauer pair** for $\mathbb{F}G$ is a pair (Q, b_Q) where Q is a p-subgroup of G and b_Q is a block of $\mathbb{F}QC_G(Q)$.

Remark. Then b_Q is admissible in G; in particular, $(b_Q)^G$ is defined. If $(b_Q)^G = B$ then (Q, b_Q) is also called a Brauer pair for B. This gives a partition of the Brauer pairs for $\mathbb{F}G$ according to the blocks of $\mathbb{F}G$. The group G acts by conjugation on the set of Brauer pairs for $\mathbb{F}G$:

$$^{g}(Q, b_Q) := (gQg^{-1}, gb_Qg^{-1}) \quad \text{for} \quad g \in G.$$

If (Q, b_Q) is a Brauer pair for B then ${}^g(Q, b_Q)$ is also a Brauer pair for B. The stabilizer

$$N_G(Q, b_Q) := \{g \in G : {}^g(Q, b_Q) = (Q, b_Q)\}$$

is called the **normalizer** of (Q, b_Q) in G.

Example. By Brauer's Third Main Theorem, the Brauer pairs for the principal block B of $\mathbb{F}G$ have the form (Q, b_Q) where b_Q is the principal block of $\mathbb{F}QC_G(Q)$. Thus the map $(Q, b_Q) \longmapsto Q$ is a bijection between the set of Brauer pairs for the principal block of $\mathbb{F}G$ and the set of p-subgroups of G. In particular, we have

$$N_G(Q, b_Q) = N_G(Q)$$

in this case.

19. Nilpotent blocks

Definition

A block B of the group algebra $\mathbb{F}G$ is called **nilpotent** if $N_G(Q, b_Q)/C_G(Q)$ is a p-group, for every Brauer pair (Q, b_Q) of B.

Example. Thus the principal block B_0 of $\mathbb{F}G$ is nilpotent iff $N_G(Q)/C_G(Q)$ is a p-group, for every p-subgroup Q of G. By a theorem of FROBENIUS, this is the case iff G is p-nilpotent (i.e. $G = SK$ and $S \cap K = 1$ where $S \in \mathrm{Syl}_p(G)$ and $K \trianglelefteq G$.) It follows that $B_0 \cong \mathbb{F}S$ in this case. For arbitrary nilpotent blocks, the following holds:

Theorem (PUIG)

Let B be a nilpotent block of $\mathbb{F}G$, with defect group D. Then $B \cong (\mathbb{F}D)^{n \times n}$, for some $n \in \mathbb{N}$.

Remark. It is an open problem to show that, conversely, a block B which is Morita equivalent to $\mathbb{F}D$, for some finite p-group D, is nilpotent (and it is also open whether, in this case, the defect groups of B are isomorphic to D). This latter question is related to the following open problem:

Modular Isomorphism Problem

Let P, Q be finite p-groups such that $\mathbb{F}P \cong \mathbb{F}Q$. Does it follow that P and Q are isomorphic?

Examples of nilpotent blocks are:

- Blocks of $\mathbb{F}G$ for a p-nilpotent group G
- Blocks of $\mathbb{F}G$ with defect group D whenever $G = DC_G(D)$

Because of Broué's abelian defect group conjecture, blocks with abelian defect groups are of special interest.

Proposition

Let B be a block of the group algebra $\mathbb{F}G$ with an abelian defect group D, and let b be a block of $\mathbb{F}DC_G(D)$ with $b^G = B$. Then B is nilpotent iff $N_G(D, b) = DC_G(D)$.

In the following, we will have to use a generalization of group algebras.

Definition

A **twisted group algebra** of a finite group G over $\mathbb{F}$ is a finite-dimensional $\mathbb{F}$-algebra A, together with a fixed decomposition

$$A = \bigoplus_{x \in G} A_x$$

into 1-dimensional subspaces A_x such that $A_x A_y = A_{xy}$ for $x, y \in G$.

If we choose a nonzero element $u_x \in A_x$, for $x \in G$, then we get equations

$$u_x u_y = \gamma(x, y) u_{xy}$$

with $\gamma(x, y) \in \mathbb{F}^\times$ for $x, y \in G$.

Associativity in A implies that

$$\gamma(x,y)\gamma(xy,z) = \gamma(x,yz)\gamma(y,z) \text{ for } x,y,z \in G.$$

Thus $\gamma : G \times G \longrightarrow \mathbb{F}^\times$ is a **2-cocycle**. Conversely, every 2-cocycle $\gamma : G \times G \longrightarrow \mathbb{F}^\times$ defines a twisted group algebra of G over $\mathbb{F}$ which we denote by $\mathbb{F}_\gamma G$. (It is easy to see that every 2-cocycle $\gamma' : G \times G \longrightarrow \mathbb{F}^\times$ which differs from γ by a 2-coboundary gives rise to an isomorphic twisted group algebra $\mathbb{F}_{\gamma'} G$.)

21. Blocks with normal defect groups

Motivated by Brauer's First Main Theorem, we consider the structure of blocks with a normal defect group:

- B block of a group algebra $\mathbb{F}G$ with defect group $D \trianglelefteq G$
- b block of $\mathbb{F}DC_G(D)$ such that $b^G = B$ [Then b has defect group D.]
- $I := I_G(b)$ [Then $p \nmid |I : DC_G(D)|$.]

Thus $\overline{D} := DC_G(D)/C_G(D)$ is a normal Sylow p-subgroup of $\overline{I} := I/C_G(D)$. We construct a group L in the following way:

By the Schur-Zassenhaus theorem, there is a subgroup $\overline{K}$ of $\overline{I}$ (unique up to conjugation) such that

$$\overline{I} = \overline{D} \cdot \overline{K} \quad \text{and} \quad \overline{D} \cap \overline{K} = 1.$$

Since $\overline{K}$ acts naturally on D, we may form the semidirect product $L = D \cdot \overline{K}$.

Theorem (Külshammer)

In the situation above, we have

$$B \cong (\mathbb{F}_\gamma L)^{n\times n},$$

for some $n \in \mathbb{N}$ and some 2-cocycle $\gamma : L \times L \longrightarrow \mathbb{F}^\times$.

Remark. This implies, for example, that Donovan's conjecture holds for blocks with normal defect groups.

- $\mathbb{F}$ algebraically closed field, $\mathrm{char}(\mathbb{F}) = p > 0$
- G finite group

Definition

An $\mathbb{F}G$-module M is called **relatively H-projective**, for a subgroup H of G, if $M \mid \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$.

Here $\mathrm{Res}_H^G(M)$ denotes the **restriction** of M to H, and $\mathrm{Ind}_H^G(N) := \mathbb{F}G \otimes_{\mathbb{F}H} N$ denotes the **induction** of an $\mathbb{F}H$-module N. There are various other characterizations of relatively H-projective modules (including HIGMAN's criterion) which we omit.

Example. (i) For $S \in \mathrm{Syl}_p(G)$, every $\mathbb{F}G$-module is relatively S-projective.

(ii) If the $\mathbb{F}G$-module M belongs to a block B of $\mathbb{F}G$ with defect group D then M is relatively D-projective.

(iii) An $\mathbb{F}G$-module M is relatively 1-projective iff M is projective.

Definition (Green)

Let M be an indecomposable $\mathbb{F}G$-module. A subgroup Q of G is called a **vertex** of M if M is relatively Q-projective but not relatively R-projective for any proper subgroup R of Q.

Then the vertices of an indecomposable $\mathbb{F}G$-module M form a conjugacy class of p-subgroups of G.

Remark. (i) The indecomposable projective $\mathbb{F}G$-modules have vertex 1.

(ii) If M is an indecomposable module $\mathbb{F}G$-module with vertex Q belonging to a block B of $\mathbb{F}G$ with defect group D then $Q \leq_G D$.

(iii) If B is a block of $\mathbb{F}G$ with defect group D then there exists a simple $\mathbb{F}G$-module L belonging to B which has vertex D.

(iv) The vertices of the trivial $\mathbb{F}G$-module are the Sylow p-subgroups of G. This is a consequence of the following more general fact:

(v) If M is an indecomposable $\mathbb{F}G$-module with vertex $Q \leq S \in \mathrm{Syl}_p(G)$ then $|S : Q| \mid \dim M$.

The proof of (v) makes use of the following important result:

GREEN's Indecomposability Theorem

Let $K \trianglelefteq G$ such that G/K is a p-group, and let N be an indecomposable $\mathbb{F}K$-module. Then $\mathrm{Ind}_K^G(N)$ is indecomposable.

23. Sources and the Green correspondence

Let M be an indecomposable $\mathbb{F}G$-module with vertex Q. Since $M \mid \mathrm{Ind}_Q^G(\mathrm{Res}_Q^G(M))$ there exists an indecomposable $\mathbb{F}Q$-module V such that $M \mid \mathrm{Ind}_Q^G(V)$ [and $V \mid \mathrm{Res}_Q^G(M)$]. One can show that V is unique up to isomorphism and $N_G(Q)$-conjugation.

Definition

In the situation above, V is called a **source** of M.

For sources of simple (not just indecomposable) $\mathbb{F}G$-modules, the following finiteness conjecture (related to DONOVAN's conjecture) is of interest:

Conjecture (FEIT)

For any finite p-group P, there are only finitely many indecomposable $\mathbb{F}P$-modules which arise as sources of simple $\mathbb{F}G$-modules for finite overgroups G of P.

DADE has proved that FEIT's conjecture holds for the indecomposable $\mathbb{F}P$-modules of any fixed dimension d. Thus it remains to show that there is an upper bound for the dimensions of the indecomposable $\mathbb{F}P$-modules which are sources of simple $\mathbb{F}G$-modules, for finite overgroups G of P.

In connection with Broué's abelian defect group conjecture indecomposable modules with *trivial* sources are especially important. As was explained by Joe Chuang, these are the modules entering into the definition of splendid Rickard complexes. Also, the indecomposable modules with trivial sources are just the components of permutation modules.

In the following, we discuss the important Green correspondence. In order to do this, we fix the following notation:

- P is a p-subgroup of a finite group G, and $N_G(P) \leq H \leq G$
- $\mathcal{X} := \{Q : Q \leq P \cap gPg^{-1}$ for some $g \in G \setminus H\}$
- $\mathcal{Y} := \{Q : Q \leq H \cap gPg^{-1}$ for some $g \in G \setminus H\}$
- $\mathcal{Z} := \{Q \leq P : Q \notin \mathcal{X}\}$

One considers $\mathcal{X}$ and $\mathcal{Y}$ as sets of bad subgroups, and $\mathcal{Z}$ as a set of good subgroups.

Theorem (Green correspondence)

(i) If M is an indecomposable $\mathbb{F}G$-module with vertex $Q \in \mathcal{Z}$ then $\mathrm{Res}_H^G(M)$ has a unique (up to isomorphism) component M' with vertex Q. It has multiplicity 1 in $\mathrm{Res}_H^G(M)$, and the other components of $\mathrm{Res}_H^G(M)$ have a vertex in $\mathcal{Y}$.

(ii) If N is an indecomposable $\mathbb{F}H$-module with vertex $Q \in \mathcal{Z}$ then $\mathrm{Ind}_H^G(N)$ has a unique (up to isomorphism) component N' with vertex Q. It has multiplicity 1 in $\mathrm{Ind}_H^G(N)$, and the other components of $\mathrm{Ind}_H^G(N)$ have a vertex in $\mathcal{X}$.

(iii) By (i) and (ii), we get mutually inverse bijections

$$\begin{gathered}\{\text{isomorphism classes of indecomposable } \mathbb{F}G\text{-modules} \\ \text{with vertex in } \mathcal{Z}\} \\ \updownarrow \\ \{\text{isomorphism classes of indecomposable } \mathbb{F}H\text{-modules} \\ \text{with vertex in } \mathcal{Z}\}\end{gathered}$$

preserving vertices and sources.

Remark. In general, Green correspondents of simple modules are not simple. However, the Green correspondence behaves nicely with respect to block induction:

Theorem (Nagao)

Let M be an indecomposable $\mathbb{F}G$-module belonging to a block B of $\mathbb{F}G$, and let $H \leq G$. Moreover, let N be a component of $\mathrm{Res}_H^G(M)$ belonging to a block b of $\mathbb{F}H$, and let Q be a vertex of N. If $C_G(Q) \subseteq H$ then $b^G = B$.

These facts can be used to show:

Theorem

Let B be a block of a group algebra $\mathbb{F}G$ with defect group D. Then B has finite representation type iff D is cyclic.

24. Weights

- $\mathbb{F}$ algebraically closed field, $\mathrm{char}(\mathbb{F}) = p > 0$
- G finite group

Definition (Alperin)

An $\mathbb{F}G$-**weight** is a pair (Q, W_Q) where Q is a p-subgroup of G and W_Q is a block of $\mathbb{F}[N_G(Q)/Q]$ of defect zero.

Remark. (i) We recall that, for every p-subgroup Q of G, there is a natural bijection

$$\{\text{blocks of } \mathbb{F}[N_G(Q)/Q] \text{ of defect } 0\}$$
$$\updownarrow$$
$$\{\text{isomorphism classes of simple projective } \mathbb{F}[N_G(Q)/Q]\text{-modules}\}$$

Moreover, inflation gives a bijection between the latter set and

$$\{\text{isomorphism classes of simple } \mathbb{F}N_G(Q)\text{-modules with vertex } Q\}.$$

(ii) Let (Q, W_Q) be an $\mathbb{F}G$-weight, and let M_Q be the corresponding simple $\mathbb{F}N_G(Q)$-module with vertex Q. If M_Q belongs to the block B_Q of $\mathbb{F}N_G(Q)$ then B_Q is admissible, so that $(B_Q)^G = B$ is defined. Then (Q, W_Q) is called a B-**weight**. In this way we get a partition of the set of $\mathbb{F}G$-weights according to the blocks of $\mathbb{F}G$.

(iii) G acts on the set of $\mathbb{F}G$-weights by conjugation. If (Q, W_Q) is a B-weight for a block B of $\mathbb{F}G$ then so is ${}^g(Q, W_Q)$, for $g \in G$.

(iv) If (Q, W_Q) is an $\mathbb{F}G$-weight then $Q = O_p(N_G(Q))$, so that Q is a radical p-subgroup of G.

(v) Let (Q, W_Q) be a B-weight, for a block B of $\mathbb{F}G$. Then the corresponding block B_Q of $\mathbb{F}N_G(Q)$ covers a block b_Q of $\mathbb{F}QC_G(Q)$ with defect group Q, and $b_Q^{N_G(Q)} = B_Q$. Then b_Q is a nilpotent block (so that $b_Q \cong (\mathbb{F}Q)^{n \times n}$ for some $n \in \mathbb{N}$), and (Q, b_Q) is a Brauer pair for B (which is unique up to conjugation).

(vi) In the situation above, the structure of the block B_Q of $\mathbb{F}N_G(Q)$ can be described by a result of KÜLSHAMMER and PUIG on so-called *extensions of nilpotent blocks*. We skip the details. (In general, Q is properly contained in a defect group of B_Q.)

Weights were introduced by ALPERIN in order to formulate one of the most important conjectures in block theory.

ALPERIN's weight conjecture, AWC

The number of isomorphism classes of simple $\mathbb{F}G$-modules, i.e. $\ell(\mathbb{F}G)$, equals the number of conjugacy classes of $\mathbb{F}G$-weights. In fact, the equality holds block by block.

Remark. This conjecture has been proved for several classes of finite groups (p-solvable groups, symmetric groups, groups of Lie type in characteristic p) and blocks, but remains open in general.

Example. For a block B of $\mathbb{F}G$ with an abelian defect group D and Brauer correspondent b in $\mathbb{F}N_G(D)$, AWC means that $\ell(B) = \ell(b)$. (This is open, but would also follow from Broué's abelian defect group conjecture.)
In this situation, AWC would also imply that $\mathrm{k}(B) = \mathrm{k}(b)$. (Here Broué's abelian defect group conjecture would have the stronger consequence $Z(B) \cong Z(b)$.)

25. Blocks with cyclic defect groups

The structure of blocks with cyclic defect groups is well understood (Dade). All of the main open problems in block theory have been settled in this case. We fix the following notation.

- B block of $\mathbb{F}G$ with cyclic defect group D
- D_1 is the unique subgroup of order p in D
- $H_1 := N_G(D_1)$ and $K_1 := C_G(D_1)$

There are blocks B_1 of $\mathbb{F}H_1$ and b_1 of $\mathbb{F}K_1$ such that $(b_1)^{H_1} = B_1$ and $(B_1)^G = B$. Both B_1 and b_1 have defect group D. Also, b_1 is nilpotent, so that

$$b_1 \cong (\mathbb{F}D)^{n\times n} \text{ for some } n \in \mathbb{N},$$

by Puig's theorem. The structure of B_1 can be derived from the Külshammer-Puig theorem on *extensions of nilpotent blocks*; in this special case, there is also an easier argument by Linckelmann. One obtains:

$$B_1 \cong (\mathbb{F}L)^{m\times m} \text{ for some } m \in \mathbb{N}$$

where L is a metacyclic group; more precisely, L is the semidirect product of the cyclic defect group D and the cyclic p'-group $N_G(D, b_D)/C_G(D)$ where b_D is a block of $\mathbb{F}DC_G(D)$ with $(b_D)^G = B$. This implies the useful fact that every indecomposable B_1-module is **uniserial**; this means that it has a unique composition series.

One can derive the structure of B from that of B_1 by using the Green correspondence between G and H_1. For a simple $\mathbb{F}G$-module L belonging to B, its Green correspondent L' is a uniserial $\mathbb{F}H_1$-module belonging to B_1, so that $L'/\mathrm{Rad}(L')$ is a simple $\mathbb{F}H_1$-module belonging to B_1. It is not difficult to show that the map

$$L \longmapsto L' \longmapsto L'/\mathrm{Rad}(L')$$

induces a bijection

$$\{\text{isomorphism classes of simple } \mathbb{F}G\text{-modules belonging to } B\}$$
$$\updownarrow$$
$$\{\text{isomorphism classes of simple } \mathbb{F}H_1\text{-modules belonging to } B_1\}.$$

More precisely,

$$\ell(B) = t := |N_G(D, b_D) : DC_G(D)|.$$

This proves AWC for blocks with cyclic defect groups. It also implies that

$$\mathrm{k}(B) = \dim Z(B) = t + \frac{|D| - 1}{t}.$$

Further analysis shows that every indecomposable projective $\mathbb{F}G$-module P belonging to B satisfies

$$\mathrm{Rad}(P)/\mathrm{Soc}(P) \simeq M \oplus N$$

where M, N are uniserial modules (possibly zero). The composition factors of M and N can be computed from a special type of graph called the **Brauer tree** of B. Details can be found, for example, in Alperin's book. It is a consequence of these results that Donovan's conjecture holds for blocks with cyclic defect groups.

And RICKARD has shown that Broué's abelian defect group conjecture holds for blocks with cyclic defect groups.

26. Chains of p-subgroups

For many recent developments, a reformulation of Alperin's weight conjecture, due to KNÖRR and ROBINSON has been fundamental. This reformulation makes use of chains of p-subgroups appearing already in earlier work of QUILLEN, WEBB and others.

In the following, we denote by $\mathcal{P}_p(G)$ the set of chains of nontrivial p-subgroups

$$\sigma : P_1 < \ldots < P_n$$

of G, including the empty chain $\emptyset$. For σ as above, $|\sigma| := n$ is called the **length** of σ; in particular, we have $|\emptyset| = 0$. Then G acts on $\mathcal{P}_p(G)$ by conjugation, and $\mathcal{P}_p(G)/G$ denotes the set of orbits $[\sigma]$. Also,

$$G_\sigma := N_G(P_1) \cap \ldots \cap N_G(P_n)$$

denotes the stabilizer of σ under this action. KNÖRR and ROBINSON prove that b^G is defined, for every block b of $\mathbb{F}G_\sigma$. For a block B of $\mathbb{F}G$, we denote by B_σ the sum of blocks b of $\mathbb{F}G_\sigma$ such that $b^G = B$.

Proposition (Knörr-Robinson)

AWC holds for all finite groups G and all blocks B of $\mathbb{F}G_\sigma$ iff

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma) = 0$$

for all finite groups G and all blocks B of $\mathbb{F}G$ of positive defect.

Here the notation $\in_G$ means that the sum ranges over a transversal for $\mathcal{P}_p(G)/G$. (The choice of the transversal is irrelevant.)

Remarks. (i) If B is a block of defect 0 then $B_\sigma = 0$ whenever $\emptyset \neq \sigma \in \mathcal{P}_p(G)$. Thus

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma) = \ell(B) = 1.$$

(ii) It is not difficult to see that

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma) = \sum_{\sigma \in_G \mathcal{E}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma)$$

where $\mathcal{E}_p(G)$ consists of the chains $\sigma : P_1 < \ldots < P_n$ in $\mathcal{P}_p(G)$ such that $P_1, \ldots, P_n$ are elementary abelian. The reason is that the summands in

$$\sum_{\sigma \in_G \mathcal{P}_p(G) \setminus \mathcal{E}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma)$$

can be shown to cancel in pairs. In fact, let $\sigma : P_1 < \ldots < P_n$ be a chain in $\mathcal{P}_p(G) \setminus \mathcal{E}_p(G)$. Then the Frattini subgroup $\Phi(P_n)$ is nontrivial. Let m be minimal such that $\Phi(P_n) \subseteq P_m$. If $\Phi(P_n)P_{m-1} = P_m$ (where we set $P_0 := 1$) then σ is paired with the chain $\sigma' \in \mathcal{P}_p(G) \setminus \mathcal{E}_p(G)$ obtained from σ by removing P_m. Then $|\sigma'| = |\sigma| - 1$, $G_{\sigma'} = G_\sigma$ and $B_{\sigma'} = B_\sigma$, so the contributions from σ and σ' to the alternating sum cancel.

(iii) In a similar way, one can show that

$$\sum_{\sigma \in_G \mathcal{N}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma) = \sum_{\sigma \in_G \mathcal{E}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma);$$

here $\mathcal{N}_p(G)$ consists of the chains $\sigma : P_1 < \ldots < P_n$ in $\mathcal{P}_p(G)$ such that $P_1, \ldots, P_n$ are normal in P_n.

(iv) Also, one can replace $\mathcal{P}_p(G)$ by $\mathcal{U}_p(G)$ where $\mathcal{U}_p(G)$ denotes the set of chains of radical p-subgroups of G.

(v) It also suffices to consider the set $\mathcal{R}_p(G)$ of **radical p-chains** $\sigma : P_1 < \ldots < P_n$. These are characterized by $P_i = O_p(G_{\sigma_i})$ for all i, where $\sigma_i : P_1 < \ldots < P_i$ denotes an **initial subchain**.

(vi) The proof of the Knörr-Robinson result makes use of (i) and proceeds by induction on $|G|$.

(vii) For a nontrivial p-subgroup Q of G, the G-orbits of chains in $\mathcal{N}_p(G)$ starting with conjugates of Q are in bijection with $\mathcal{N}_p(N_G(Q)/Q)/[N_G(Q)/Q]$. Thus, if $\mathcal{Q}$ denotes a transversal for the conjugacy classes of nontrivial p-subgroups of G, we get

$$\sum_{\sigma \in_G \mathcal{N}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma) = \ell(B) - \sum_{Q \in \mathcal{Q}} S_Q$$

where S_Q is a similar alternating sum for a block b_Q of $\mathbb{F}[N_G(Q)/Q]$ (or rather a sum of such blocks). Then $S_Q = 0$, by induction, whenever b_Q has positive defect, and $S_Q = 1$ whenever b_Q has defect zero. So we see that $\sum_{Q \in \mathcal{Q}} S_Q$ is just the number of B-weights.

This proves the Knörr-Robinson result.

Instead of varying the index set of the alternating sum, one can also vary the function ℓ:

Proposition (Knörr-Robinson)

If B is a block of $\mathbb{F}G$ then

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \ell(B_\sigma) = \sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \ell_0(B_\sigma)$$

$$= \sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \mathrm{k}(B_\sigma).$$

Here, as before, $\mathrm{k}(B) = \dim Z(B)$. Moreover, $\ell_0(B)$ denotes the multiplicity of 1 as an elementary divisor of the Cartan matrix C of B. This is the same as the rank of the matrix $\overline{C}$ which is obtained from C by reducing all entries mod p. These numbers are related by

$$\ell_0(B) \leq \ell(B) \leq \mathrm{k}(B).$$

BOLTJE has shown that one can also replace $\mathrm{k}(B_\sigma)$ by $\mathrm{t}(B_\sigma)$, the number of isomorphism classes of indecomposable modules with a trivial source belonging to B_σ. And SYMONDS has worked with functions counting "multiplicities of lower defect groups". Also, in these alternating sums one can replace $\mathcal{P}_p(G)$ by any of the other sets of chains of p-subgroups mentioned above.

In the following, we are going to discuss refinements of the Knörr-Robinson version of Alperin's weight conjecture. These refinements started with DADE, but more recently included other people like NAVARRO, ISAACS, UNO, TURULL and perhaps others.

In order to explain these refinements, we need to change our point of view.

27. Valuation rings

In the following, we will look at the following situation:

- G a finite group
- $\mathcal{O}$ a complete discrete valuation ring of characteristic 0
- $\wp$ the unique maximal ideal of $\mathcal{O}$
- $\mathbb{F} = \mathcal{O}/\wp$ algebraically closed field, $\mathrm{char}(\mathbb{F}) = p > 0$
- $\zeta \in \mathcal{O}$ a primitive $|G|$-th root of unity
- $\mathbb{K}$ field of fractions of $\mathcal{O}$

Then $(\mathbb{K}, \mathcal{O}, \mathbb{F})$ is called a **splitting** p**-modular system** for G. Thus $\mathcal{O}G \subseteq \mathbb{K}G$, and we have a natural map $\mathcal{O}G \longrightarrow \mathbb{F}G$. We denote the set of irreducible characters of $\mathbb{K}G$ by $\mathrm{Irr}(\mathbb{K}G)$. Now Wedderburn's theorem implies that

$$\mathbb{K}G \cong \prod_{\chi \in \mathrm{Irr}(\mathbb{K}G)} \mathbb{K}^{\chi(1) \times \chi(1)}.$$

Let $B_1, \dots, B_r$ denote the blocks of $\mathbb{F}G$, and let $e_1, \dots, e_r$ denote the corresponding block idempotents. Then $e_1, \dots, e_r$ lift uniquely to (block) idempotents $\hat{e}_1, \dots, \hat{e}_r \in Z(\mathcal{O}G)$. We set $\hat{B}_i := \mathcal{O}G\hat{e}_i = \hat{e}_i\mathcal{O}G$ for $i = 1, \dots, r$. Then we get a block decomposition

$$\mathcal{O}G = \hat{B}_1 \oplus \cdots \oplus \hat{B}_r.$$

For $\chi \in \mathrm{Irr}(\mathbb{K}G)$, there is a unique $i \in \{1, \dots, r\}$ such that $\chi(\hat{e}_i) \neq 0$. We set

$$\mathrm{Irr}(\hat{B}_i) := \{\chi \in \mathrm{Irr}(\mathbb{K}G) : \chi(\hat{e}_i) \neq 0\}.$$

Then

$$\mathbb{K}\hat{B}_i := \mathbb{K} \otimes_{\mathcal{O}} \hat{B}_i \cong \prod_{\chi \in \mathrm{Irr}(\hat{B}_i)} \mathbb{K}^{\chi(1)\times\chi(1)};$$

in particular,

$$\mathrm{k}(B_i) = \dim Z(B_i) = \mathrm{rk}_{\mathcal{O}} Z(\hat{B}_i) = \dim Z(\mathbb{K}\hat{B}_i) = |\mathrm{Irr}(\hat{B}_i)|.$$

Moreover, the irreducible characters of $\mathbb{K}G$ are partitioned according to the blocks $B_1, \ldots, B_r$ of $\mathbb{F}G$:

$$\mathrm{Irr}(\mathbb{K}G) = \bigsqcup_{i=1}^{r} \mathrm{Irr}(\hat{B}_i).$$

Recall that $\chi(1) \mid |G|$, for $\chi \in \mathrm{Irr}(\mathbb{K}G)$. We write

$$\frac{|G|}{\chi(1)} = p^{\mathrm{d}(\chi)}\mathrm{r}(\chi) \text{ where } p \not| \mathrm{r}(\chi).$$

Then $\mathrm{d}(\chi)$ is called the **(p-)defect** of χ.

Proposition

If B is a block of $\mathbb{F}G$ of defect d then

$$d = \max\{\mathrm{d}(\chi) : \chi \in \mathrm{Irr}(\hat{B})\}.$$

28. Dade's conjectures

Recall that the Knörr-Robinson reformulation of Alperin's weight conjecture asserts that

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \mathrm{k}(B_\sigma) = 0,$$

for every finite group G and every block B of $\mathbb{F}G_\sigma$ of positive defect. Here $\mathrm{k}(B_\sigma)$ is interpreted as the number of irreducible characters of $\mathbb{K}G_\sigma$ belonging to B_σ. In the following, we are going to consider refinements of this conjecture. The general shape of these refined conjectures is

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \mathrm{k}(B_\sigma, *) = 0,$$

for the same blocks as above. Here $(*)$ stands for certain properties that can be attached to irreducible characters of the stabilizers G_σ, and $\mathrm{k}(B_\sigma, *)$ denotes the number of irreducible characters of $\mathbb{K}G_\sigma$ belonging to a block in B_σ and satisfying the property $(*)$.

The best known example of such a refined conjecture is the following one:

DADE's ordinary conjecture, DOC

$$\sum_{\sigma \in_G \mathcal{P}_p(G)} (-1)^{|\sigma|} \mathrm{k}(B_\sigma, d) = 0,$$

for every finite group G, every block B of $\mathbb{F}G$ of positive defect, and every nonnegative integer d.

Here one is counting the irreducible characters of a fixed defect d.

In the following, I will try to explain a similar refinement, also due to DADE. This refinement concerns the following situation:

- Z is a central p-subgroup of G (possibly $Z = 1$)
- $\lambda \in \mathrm{Irr}(\mathbb{K}Z)$

Note that Z will be contained in every stabilizer G_σ appearing in (DOC). Thus it makes sense to count only irreducible characters χ of G_σ **lying over** λ, i.e.

$$\chi|Z = \chi(1) \cdot \lambda.$$

A combination of these two ideas (fixing a defect and fixing an irreducible character of a central p-subgroup) leads to a variant of the conjecture which can be expressed formally in the following way:

DADE's projective conjecture, DPC

$$\sum_{\sigma \in_G \mathcal{P}_p(G,Z)} (-1)^{|\sigma|} \mathrm{k}(B_\sigma, d, \lambda) = 0,$$

for every finite group G, every block B of $\mathbb{F}G$ whose defect groups properly contain Z, every nonnegative integer d and every irreducible character λ of a central p-subgroup Z of G.

Here $\mathcal{P}_p(G, Z)$ denotes the set of preimages in G of chains in $\mathcal{P}_p(G/Z)$.

Further refinements lead to forms of the conjecture carrying the following names:

- Dade's invariant conjecture
- Dade's adjoint conjecture
- Dade's inductive conjecture
- ...

These refinements all have to do with Clifford theory, i.e. with representation theory in the presence of normal subgroups. They are concerned with a situation where our group G is a normal subgroup of another group H. We omit the details.

These refined conjectures were formulated by Dade with the aim of having a form of the conjecture which reduces to simple groups. Such a reduction was announced by Dade in his paper in 1992 (and even before in a series of MSRI lectures in 1990, I think), but did not appear until this date. In the meantime other mathematicians have used the opportunity to formulate further refinements, but in a different direction.

Most likely, each of these further refinements will make the proof of a reduction to simple groups more complicated.

Almost all of the more recent refinements of Dade's conjectures first occured in a different context, namely the Alperin-McKay conjecture, so we will now discuss this conjecture.

29. The Alperin-McKay conjecture

This is a conjecture which predates Alperin's weight conjecture. The simplest form of this conjecture is the following one:

McKay conjecture

Let G be a finite group, let p be a prime number, and let $S \in \mathrm{Syl}_p(G)$. Then $m_p(G) = m_p(N_G(S))$ where $m_p(G)$ denotes the number of irreducible characters of G whose degree is not divisible by p.

Thus, for $p = 2$, the conjecture asserts that G and the normalizer in G of a Sylow 2-subgroup have the same number of irreducible characters of odd degree.

I will not say much about this conjecture since there will be a separate talk by Navarro on this topic. However, I will point out the connection with Dade's conjecture. Before I do this, I will state the block version of the McKay conjecture.

ALPERIN-MCKAY conjecture

Let B be a block of a group algebra $\mathbb{F}G$ with defect group D, and let b the Brauer correspondent of B in $N_G(D)$. Then

$$\mathrm{k}_0(B) = \mathrm{k}_0(b).$$

Here $\mathrm{k}_0(B)$ denotes the number of irreducible characters of **height zero** in B, i.e. of the same defect as B.

DADE has proved that the Alperin-McKay conjecture (in a slightly stronger form even) is implied by his projective conjecture. Thus DPC is a common generalization of both the Alperin-McKay conjecture and Alperin's weight conjecture.

30. Recent refinements

As I said before, these refinements of Dade's conjectures were first considered in connection with the Alperin-McKay conjecture where they take the form

$$\mathrm{k}_0(B,*) = \mathrm{k}_0(b,*),$$

where B is a block of a group algebra $\mathbb{F}G$ with defect group D and b is the Brauer correspondent of B in $N_G(D)$.

For simplicity, I will discuss the refinements only in the Alperin-McKay context and not in the Dade context.

The Isaacs-Navarro refinement

Recall that the degree of an irreducible character χ of G can be written in the form

$$\chi(1) = p^{\mathrm{d}(\chi)}\mathrm{r}(\chi) \text{ where } p \nmid \mathrm{r}(\chi).$$

Then Isaacs and Navarro conjecture the following:

Isaacs-Navarro conjecture

Let B be a block of a group algebra $\mathbb{F}G$ with defect group D, let b be the Brauer correspondent of B in $N_G(D)$, and let $r \in \mathbb{N}$. Then

$$\mathrm{k}_0(B, [r]) = \mathrm{k}_0(b, [r]).$$

Here $\mathrm{k}_0(B, [r])$ denotes the number of irreducible characters χ of height zero belonging to B such that

$$\mathrm{r}(\chi) \equiv \pm r \pmod{p}.$$

Note that one cannot remove the sign here.

Navarro's refinement

This refinement has to do with actions of Galois groups on irreducible characters. More details will be given in NAVARRO's talk.

Turull's refinement

This refinement also takes fields of values of irreducible characters and Schur indices over $\mathbb{Q}_p$ into account; here $\mathbb{Q}_p$ denotes the field of p-adic numbers. Again, details will be given elsewhere.

UNO has formulated refinements of Dade's conjectures which also take the ideas of ISAACS-NAVARRO and NAVARRO into account.

31. Boltje's version

Boltje has formulated refinements of Dade's conjectures in a different direction. These refinements have the form

$$\sum_{\sigma}(-1)^{|\sigma|-n}\mathrm{k}(B_\sigma,*) \geq 0$$

where the sum ranges over chains σ of length $|\sigma| \leq n$ only. This form of the conjecture is motivated by the idea that the alternating sums in Dade's conjectures should really be Euler characteristics of acyclic chain complexes.

Summary

BAD NEWS:

Whereas the

counting conjectures

take a more and more precise form, we seem to be far away from a

structural explanation.

GOOD NEWS:

Many theorems are waiting to be discovered by young people.