Broué's Abelian Defect Group Conjecture

Joe Chuang

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1 Block theory

Let G be a finite group, k an algebraically closed field. We have the block decomposition

$$kG = B_1 \oplus \cdots \oplus B_r$$

The trivial module k belongs to the principal block B_1.

Let us assume that k has characteristic $p > 0$. A defect group of B is a minimal subgroup $D \leqslant G$ such that every B-module is a summand of a kG-module induced from D. Then D is a p-subgroup, unique up to conjugacy.

We have $D = 1$ if and only if B is a matrix algebra. For the principal block B_1, the defect group D is a Sylow p-subgroup.

Example 1.1. $G = \mathfrak{S}_3$.

If $p > 3$, then $k\mathfrak{S}_3 = k \oplus k \oplus k^{2\times 2}$.

If $p = 2$, then $k\mathfrak{S}_3 = k[x]/(x^2) \oplus k^{2\times 2}$.

If $p = 3$, then $k\mathfrak{S}_3 = B_1$.

2 Local representation theory

The idea of local representation theory is to relate the category kG-mod to the categories of representations of local subgroups:

$$\{N_G(Q)\text{-mod} \mid p\text{-subgroups } Q \neq 1\}$$

For example, Brauer's First Main Theorem says that we have a bijection

$$\{\text{Blocks of } kG \text{ with defect group } D\} \simeq \{\text{Blocks of } kN_G(D) \text{ with defect group } D\}$$

Besides, principal blocks correspond through this bijection (this is Brauer's Third Main Theorem).

Another example is Alperin's weight conjecture. Let us just state a special case: if D is abelian, then $\ell(B) = \ell(C)$, where ℓ denotes the number of simple modules, and C is the block corresponding to B.

Let us now come to Broué's abelian defect group conjecture.

Conjecture 2.1. *With the above notation, if D is abelian, then B and C are derived equivalent, that is, $D(B) \simeq D(C)$.*

Derived equivalence is a notion weaker that Morita equivalence. We have the implications:

$$B \simeq C \Longrightarrow B \sim_{\mathrm{Mor}} C \Longrightarrow B \sim_{\mathrm{der}} C$$

3 Categories of complexes

We are going to briefly describe three categories of complexes: $C(B)$, $K(B)$ and $D(B)$. For all of these, the objects are the same: bounded complexes of finitely generated B-modules

$$X = \cdots X^{-1} \xrightarrow{\partial} X^0 \xrightarrow{\partial} X^1 \cdots$$

with $\partial^2 = 0$.

The morphisms in $C(B)$ are chain maps: a chain map from X to Y is a family of maps $f^i : X^i \to Y^i$ making all the squares commute.

In $K(B)$, we invert all homotopy equivalences. A complex X becomes isomorphic to 0 in $K(B)$ if and only if it is contractible, that is, if and only if X is the direct sum of complexes of the form $0 \to M \xrightarrow{\sim} M \to 0$.

In $D(B)$, we invert all quasi-isomorphisms. Here we have $X \simeq 0$ if and only if X is acyclic, that is, $H^*(X) = 0$. We will not go into technical details.

The categories $K(B)$ and $D(B)$ are triangulated categories. One can define the Grothendieck group of such a category, similar to the Grothendieck group of an abelian category. It is a free abelian group of rank $\ell(B)$. Therefore, $\ell(B)$ is invariant by derived equivalence.

To illustrate the elegance of the language of derived categories, let us give the following definition of a defect group, due to Rouquier.

Definition 3.1. A defect group of a block B of kG is a minimal subgroup of G such that $\mathrm{Res} : D(B) \to D(kD)$ is faithful.

Example 3.2. Let $p = 2$, $G = \mathfrak{A}_5$. Let P be a Sylow p-subgroup of G. Then $P \simeq C_2 \times C_2$, and $N_G(P) \simeq \mathfrak{A}_4$. Let $B = B_1(k\mathfrak{A}_5)$ and $C = B_1(k\mathfrak{A}_4) = k\mathfrak{A}_4$.

Block	Simples	Dimension	Cartan matrix
B	k, S, T	44	$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$
C	k, U, V	12	$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

The blocks B and C have the same number of simple modules. Since the two algebras have different dimensions, they cannot be isomorphic. They are not either Morita equivalent, because they do not have the same Cartan matrix. But they are derived equivalent: Rickard has constructed an equivalence $D(B) \simeq D(C)$, where

$$\begin{array}{rcl} k & \mapsto & \left(0 \to 0 \to k \to 0\right) \\ S & \mapsto & \left(0 \to \begin{smallmatrix} k \\ U \end{smallmatrix} \to 0 \to 0\right) \\ T & \mapsto & \left(0 \to \begin{smallmatrix} k \\ V \end{smallmatrix} \to 0 \to 0\right) \end{array}$$

4 Functors

In each of the following three situations, we have a pair of adjoint functors:

	F	X
$H \leqslant G$	Res : kG-mod $\longrightarrow kH$-mod Ind : kH-mod $\longrightarrow kG$-mod	${}_{kH}kG_{kG}$ ${}_{kG}kG_{kH}$
$N \trianglelefteq G, p \nmid \lvert N\rvert$	Fix : kG-mod $\longrightarrow kG/N$-mod Inf : kG/N-mod $\longrightarrow kG$-mod	${}_{kG/N}(kG/N)_{kG}$ ${}_{kG}(kG/N)_{kG/N}$
B block of G	pr : kG-mod $\longrightarrow B$-mod inc : B-mod $\longrightarrow kG$-mod	${}_{B}kG_{kG}$ ${}_{kG}kG_{B}$

All are exact and take projectives to projectives. So they are of the form $F = X \otimes_\Gamma -$ for some ${}_\Lambda X_\Gamma$ such that ${}_\Lambda X$ and X_Γ are finitely generated projectives.

A symmetric algebra is a finite dimensional algebra Λ with $\alpha : \Lambda \to k$ such that $\alpha(xy) = \alpha(yx)$ and $\operatorname{Ker}(\alpha)$ contains no nontrivial left or right ideals.

Example 4.1. kG (or a block B of G) is a symmetric algebra with $\alpha(\sum_g \lambda_g g) = \lambda_1$.

Proposition 4.2. *Let Λ be a finite dimensional algebra. Then the following assertions are equivalent:*

1. *Λ is a symmetric algebra.*
2. *$\Lambda \simeq \Lambda^* = \operatorname{Hom}_k(\Lambda, k)$ as Λ-Λ-bimodule.*
3. *$\operatorname{Hom}_\Lambda(M, \Lambda) \simeq M^*$ as right Λ-modules, natural in $M \in A$-mod.*
4. *$\operatorname{Hom}_\Lambda(M, P) \simeq \operatorname{Hom}_\Lambda(P, M)^*$ natural for P finitely generated projective Λ-module, $M \in \Lambda$-mod.*
5. *$\operatorname{Hom}_{D(\Lambda)}(M, P) \simeq \operatorname{Hom}_{D(\Lambda)}(P, M)^*$, natural for P a bounded complex of finitely generated projectives, $M \in D(\Lambda)$.*

The last two characterizations of symmetric algebras are due to Rickard. We get the following corollary:

Corollary 4.3. *If Λ and Γ are derived equivalent then*

$$\Lambda \textit{ symmetric} \Longrightarrow \Gamma \textit{ symmetric}$$

From now on, assume that Λ *and* Γ *are symmetric.* Let ${}_\Lambda X_\Gamma$ be a bimodule with ${}_\Lambda X$ and X_Γ projective. Then $X \otimes_\Gamma -$ is left and right adjoint to $X^* \otimes_\Lambda -$.

Theorem 4.4. *Λ and Γ are Morita equivalent if and only if there is X such that $X \otimes_\Gamma X^* \simeq {}_\Lambda\Lambda_\Lambda$ and $X^* \otimes_\Gamma X \simeq {}_\Gamma\Gamma_\Gamma$.*

Let now X be a bounded complex of finitely generated Λ-Γ-bimodules, projective for Λ and Γ. Then $X \otimes_\Gamma - : C(\Gamma) \to C(\Lambda)$ is left and right adjoint to $X^* \otimes_\Lambda - : C(\Lambda) \to C(\Gamma)$, and similarly with C replaced by K or D.

Theorem 4.5 (Rickard)**.** *Λ and Γ are derived equivalent if and only if there exists X such that*

$$X \otimes_\Gamma X^* \simeq {}_\Lambda\Lambda_\Lambda \oplus \textit{(acyclic)}$$

in $C(\Lambda \otimes \Lambda^{op})$ and

$$X^* \otimes_\Gamma X \simeq {}_\Gamma\Gamma_\Gamma \oplus \textit{(acyclic)}$$

in $C(\Gamma \otimes \Gamma^{op})$.

Then X is called a two-sided tilting complex.

5 Compatible equivalences

Let G be a finite group. Let P be a Sylow p-subgroup of G. Assume that P is abelian. Let $H = N_G(P)$. Recall that Broué's ADGC says that the principal blocks of kG and kH should be derived equivalent.

For $P \leqslant L \leqslant G$, we should have a derived equivalence

$$D(B_1(kL)) \sim_{\text{der}} D(B_1(k(H \cap L)))$$

Can we ask for some compatibility conditions ?

Let us consider the important particular case where $L = C_G(Q)$, $Q \leqslant P$, $N_L(P) = C_H(Q)$.

Brauer's construction

Γ finite group, $R \leqslant \Gamma$.

$$\begin{array}{ccc} \Gamma\text{-set} & \longrightarrow & N_\Gamma(R)\text{-set} \\ \Omega & \longmapsto & \Omega^R \end{array}$$

The linear version

$$\begin{array}{ccc} \Gamma\text{-perm} & \longrightarrow & N_\Gamma(R)\text{-perm} \\ k\Omega & \longmapsto & k\Omega^R \end{array}$$

is not well-defined.

But if R is a p-subgroup

$$k\Omega^R = \frac{(k\Omega)^R}{\sum_{Q<R} \mathrm{Tr}_Q^R((k\Omega)^Q}$$

extends to

$$\begin{array}{ccc} k\Gamma\text{-mod} & \longrightarrow & kN_\Gamma(R)\text{-mod} \\ M & \longmapsto & M(R) \end{array}$$

Let $B = B_1(k\Gamma)$ and $C = B_1(kH)$ as above. If B and C are derived equivalent, then there exists ${}_BX_C$ a two-sided tilting complex: each term is finitely generated projective over B and C, and

$$X \otimes_C X^* \simeq {}_BB_B \oplus E$$
$$X^* \otimes_B X \simeq {}_CC_C \oplus E'$$

where E and E' are acyclic.

Then X is a $G \times H$-module, with $(g, h).x = gxh^{-1}$.

Let $Q \leqslant P$.

$X(\Delta Q)$ $N_{G\times H}(\Delta Q) \supset C_G(Q) \times C_H(Q)$-module

Actually a B_Q-C_Q-bimodule, where $B_Q = B_1(kC_G(Q))$ and $C_Q = B_1(kC_H(Q))$.

$$(X \otimes_C X^*)(\Delta Q) \simeq B(\Delta Q) \oplus E(\Delta Q)$$

where $B(\Delta Q) = B_Q$. But $E(\Delta Q)$ may not be acyclic. However, it is contractible if E is.

Definition 5.1. X is a Rickard complex if E and E' are contractible. Then $X \otimes_C -$ induces $K(C) \simeq K(B)$ is a Rickard equivalence.

Is $(X \otimes_C X^*)(\Delta Q) \simeq X(\Delta Q) \otimes_{CQ} X(\Delta Q)^*$? Yes if X is splendid:

Definition 5.2. X is splendid if each term of X is a direct sum of summands of $kG \otimes_R kH$ where $R \leqslant P$.

Theorem 5.3. *If ${}_BX_C$ is a splendid Rickard complex, then ${}_{B_Q}X(\Delta Q)_{C_Q}$ is as well, $Q \leqslant P$.*

We can now state Rickard's refinement of ADGC.

Conjecture 5.4. *If G has abelian p-Sylow subgroups, then the principal blocks $B_1(kG)$ and $B_1(kN_G(P))$ are splendidly Rickard equivalent.*

Remark 5.5. A two-sided tilting complex can always be replaced by a quasi-isomorphic Rickard complex, but splendidness is not preserved by quasi-isomorphisms.

Remark 5.6. Splendidness can be formulated for any finite groups G and H with a common Sylow p-subgroup (not necessarily abelian); there are versions for non-principal blocks (Harris, Linckelmann, Puig).

Remark 5.7. Rouquier showed that, given local splendid Rickard equivalences, one can glue them together to give a global stable equivalence. We will not define this notion: let us just say that it is an equivalence modulo projectives.

This is an inductive approach to ADGC. Remaining problem: lift stable equivalences to splendid Rickard equivalences.

Remark 5.8. For finite reductive groups, X should be come from a Deligne-Lusztig variety.

6 Characters

Take a complete discrete valuation ring $\mathcal{O}$ with quotient field K of characteristic 0 and residue field k of characteristic p. Assume that k and K are large.

If M is an $\mathcal{O}$-module, we set $kM := k \otimes_{\mathcal{O}} M$ and $KM = K \otimes_{\mathcal{O}} M$.

We have block decompositions

$$kG = B_1(kG) \oplus \cdots \oplus B_r(kG)$$
$$\mathcal{O}G = B_1(\mathcal{O}G) \oplus \cdots \oplus B_r(\mathcal{O}G)$$

with $kB_i(\mathcal{O}G) \simeq B_i(kG)$.

Let us now state Broué's ADGC over $\mathcal{O}$.

Conjecture 6.1. *Let B be a block of $\mathcal{O}G$ with abelian defect group D. Let C be the Brauer correspondent block of $\mathcal{O}N_G(D)$. Then $D(B) \simeq D(C)$.*

If ${}_BX_C$ is a two-sided tilting complex over $\mathcal{O}$, then ${}_{kB}kX_{kC}$ (resp. ${}_{KB}KX_{KC}$ is a two-sided tilting complex over k (resp. over K).

In $D(KB)$ and $D(KC)$, every indecomposable object is a shifted simple. If we have an equivalence $D(B) \simeq D(C)$, passing to Grothendieck groups we get an isometry $\mathrm{ch}(B) \simeq \mathrm{ch}(C)$ between groups of virtual characters restricting to an isomorphism of the subgroups of virtual projective characters (the virtual characters which vanish on elements of order divisible by p). Broué calls this a perfect isometry. One gets a signed bijection between irreducible characters.

Theorem 6.2. *Splendid Rickard equivalences have unique lifts from k to $\mathcal{O}$, and induce isotypies between B and C, that is, a compatible family of local perfect isometries.*

7 Non-abelian defect groups

Let us recall the statement of Broué's ADGC. If B is a block of kG, with abelian defect group P, and if C is its Brauer correspondent block of $kN_G(P)$, then $D(B) \simeq D(C)$.

If P is not abelian, we might have B-weights (Q, V) for $Q < P$, and thus $\ell(C) < \ell(B)$.

Gluing problem: we need to replace $D(C)$ by some triangulated category built from blocks of local subgroups, that is, blocks b of $kN_G(Q)$, with $1 < Q \leqslant P$, such that $b^G = B$ (built from $(\mathcal{F}_P(B), \alpha)$).

To side step this problem, let us reformulate ADGC for the case of principal blocks, for the sake of simplicity.

Let G and H be two finite groups with a common Sylow p-subgroup P. If $\mathcal{F}_P(G) = \mathcal{F}_P(H)$ and P is abelian, then we have a derived equivalence of the principal blocks of G and H.

Let us explain why this is equivalent to ADGC. By Burnside's theorem, if P is abelian, then $\mathcal{F}_P(G) = \mathcal{F}_P(N_G(P))$, so this implies ADGC. In the other direction, we have $B_1(kN_G(P)) \simeq kP \rtimes E$, where $E = N_G(P)/C_G(P) = \mathrm{Aut}_{\mathcal{F}_P(G)}(P)$.

Remark 7.1. Here is a counterexample for P nonabelian: take $p = 2$, $G = \mathrm{Sz}(8)$, $H = N_G(P)$. Then $\mathcal{F}_P(G) = \mathcal{F}_P(H)$ but $D(B_1(kG)) \not\simeq D(B_1(kH))$. However $B_1(kG)$ and $B_1(kH)$ are stably equivalent.

Conjecture 7.2 (Auslander)**.** *Stably equivalent finite dimensional algebras have the same number of non-projective simple modules.*

Remark 7.3 (Rickard). Let G_i and H_i have a common Sylow p-subgroup P_i, for $i = 1, 2$. Let $G = G_1 \times G_2$ and Let $G = H_1 \times H_2$ and $P = P_1 \times P_2$. Assume that $\mathcal{F}_{P_i}(G_i) = \mathcal{F}_{P_i}(H_i)$. Then $\mathcal{F}_P(G) = \mathcal{F}_P(H)$.

We have $B_1(kG) \simeq B_1(kG_1) \otimes B_1(kG_2)$ and $B_1(kH) \simeq B_1(kH_1) \otimes B_1(kH_2)$.

If Γ_i and Λ_i are derived equivalent, then $\Gamma_1 \otimes \Gamma_2$ and $\Lambda_1 \otimes \Lambda_2$ are derived equivalent. But this is not clear for stable equivalences.

Remark 7.4 (Benson). In the $G = \mathrm{Sz}(8)$, $p = 2$ example. A perfect isometry is an isometry $\mathbb{Z}\,\mathrm{Irr}(B_1(\mathcal{O}G)) \simeq \mathbb{Z}\,\mathrm{Irr}(B_1(\mathcal{O}H))$ with additional properties. There is no perfect isometry over $\mathbb{Z}$ is this case, but Benson observed that there is one over $\mathbb{Z}\left[\frac{1}{2}\right]$.

Instead of getting a signed bijection, one might associate to a character χ a linear combination such as $\frac{1}{2}(\psi_1 - \psi_2 - \psi_3 - \psi_4)$.

Remark 7.5. There are many examples of derived equivalences between blocks of non-abelian defect groups (Koshitani, Kunugi, Usami. . .).

8 Symmetric groups

Let us consider the symmetric S_n. Let w be an positive integer such that $n - pw < p$. Then we can choose a Sylow p-subgroup P_w of S_n in S_{pw}.

Theorem 8.1 (Puig)**.** *Let B be a block of kS_n. Then P_w is a defect group of B for some $w = wt(B)$, $pw \leqslant n$, and $\mathcal{F}_{P_w}(B) = \mathcal{F}_{P_w}(S_{pw})$.*

Theorem 8.2 (Chuang-Rouquier)**.** *Let B and B' be two blocks of kS_n and $kS_{n'}$. Then $D(B) \simeq D(B')$ if and only if $wt(B) = wt(B')$.*

P_w is abelian if and only if $w < p$.

Theorem 8.3 (Chuang-Kessar)**.** *If $w < p$ then there is a block B with $wt(B) = w$, called a Rouquier block, such that B-mod $\simeq B_1(S_p \wr S_w)$*

Combining with results of Rickard, Linckelmann, Rouquier, and Marcus, one can conclude that ADGC holds for symmetric groups.

Let us now come to the non-abelian case. Will Turner constructs algebras, which he calls $D_{A_{p-1}}(w, w)$, defined for all $w \geqslant 0$, equipped with an idempotent e_w, such that:

- we have
$$e_w D_{A_{p-1}}(w, w) e_w \simeq B_1(kS_p \wr S_w)$$
- if $w < p$ then $e_w = 1$

- $D_{A_{p-1}}(w,w)$ is conjecturally Morita equivalent to some block (a Rouquier block) of weight w.

He came close to prove this: both have the same decomposition numbers, and they are equipped with filtrations such that their graded are isomorphic.

He considers

$$\bigoplus_{w \geqslant 0} D_{A_{p-1}}(w,w) = B \otimes B^*$$

where B is a bialgebra, a generalized Schur algebra (a sum of Schur algebras of various degrees). This is endowed with a funny product.

One can replace the Dynkin diagram A_{p-1} by any graph.

One still has to make a connection with $\mathcal{F}_{P_w}(B)$.

Let us now give an idea of the proof of Theorem 8.2. We consider all blocks of symmetric groups simultaneously.

The category

$$\mathcal{F} = \bigoplus_{n \geqslant 0} kS_n\text{-mod}$$

is endowed with restriction and induction functors Res and Ind. The Jucys-Murphy elements $X_n = (1,n) + \cdots + (n-1,n)$ act on $\mathrm{Res}^{S_n}_{kS_{n-1}}$. Splitting into characteristic spaces, we get decompositions $\mathrm{Res} = E_0 + \cdots + E_{p-1}$ and $\mathrm{Ind} = F_0 + \cdots + F_{p-1}$.

Theorem 8.4 (Ariki, Grojnowski, Lascoux-Leclerc-Thibon)**.** *The action of* e_0, ..., e_{p-1}, f_0, ..., f_{p-1} *on*

$$\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{F}) = \bigoplus_{B \ block} \mathbb{C} \otimes_{\mathbb{Z}} K_0(B)$$

extends to an action of $\widehat{\mathfrak{sl}}_p(\mathbb{C})$*, the decomposition into blocks being the weight space decomposition.*

The affine Weyl group $W = \langle s_0, \ldots, s_{p-1} \rangle$ *acts transitively on blocks* B *with fixed weight* $w = wt(B)$*.*

Using ideas of Ariki, Cabanes-Rickard, Grojnowski, Grojnowski-Vazirani, Kleshchev, Lascoux-Leclerc-Thibon, Rickard, ... we can lift the action of

$$\exp(-f_i)\exp(e)\exp(-f_i)$$

to an equivalence $D(\mathcal{F}) \simeq D(\mathcal{F})$.