Bijections Towards Counting Conjectures in Finite Reductive Groups

Paul Fong
(joint work with Michel Broué and Bhama Srinivasan)

University of Illinois at Chicago

Berkeley, February 2008

The Brauer context:

- $G^\vee = \coprod_B B$, a partition into ℓ-blocks B.
- Brauer induction $b \mapsto b^G$ maps blocks of H to blocks of G whenever $DC_G(D) \leq H \leq N_G(D)$ for an ℓ-subgroup D.
- A defect group of B is a maximal D such that $B = b^G$ for a block b of $N_G(D)$. Defect groups of B are G-conjugate.
- Brauer induction induces a bijection between blocks of G with defect group D and blocks of $N_G(D)$ with defect group D.

Suppose

B ℓ-block of G with defect group D of order ℓ^a

b Brauer correspondent of B in $N_G(D)$

The integers

$$s(\chi) = |G|/\chi(1) \text{ for } \chi \in B$$
$$s(\psi) = |N_G(D)|/\psi(1) \text{ for } \psi \in b$$

are called the codegrees or Schur elements of χ and ψ.

$s(\chi)_\ell$ divides ℓ^a, $\quad s(\psi)_\ell$ divides ℓ^a.

Fix integers δ, r, where $(\ell, r) = 1$. Define

$$\begin{aligned} B_\delta &= \{\chi \in B : s(\chi)_\ell = \ell^\delta\} \\ B_{\delta,r} &= \{\chi \in B_\delta : s(\chi)_{\ell'} \equiv \pm r \pmod{\ell}\} \end{aligned}$$

Alperin-McKay Conjecture

$$|B_a| = |b_a|, \qquad (|D| = \ell^a)$$

Isaacs-Navarro Conjecture

$$|B_{a,r}| = |b_{a,r}|, \qquad (|D| = \ell^a)$$

Counts involving Galois groups and Schur Indices:

Let $\mathcal{G}_\ell = \mathrm{Gal}(\mathbf{Q}_\ell(\sqrt[|G|]{1})/\mathbf{Q}_\ell)$, $\mathbf{Q}_\ell$ is the ℓ-adic completion of $\mathbf{Q}$.

Let $m_\ell(\chi)$ be the Schur index of a character χ over $\mathbf{Q}_\ell$.

For any $\mathcal{H} \leq \mathcal{G}_\ell$ and any positive integer m

$$\begin{aligned} B_{\delta,\mathcal{H}} &= \{\chi \in B_\delta : {}^\sigma\chi = \chi \text{ for } \sigma \in \mathcal{H}\} \\ B_{\delta,m} &= \{\chi \in B_\delta : m_\ell(\chi) = m\} \end{aligned}$$

Navarro Conjecture

$$|B_{a,\mathcal{H}}| = |b_{a,\mathcal{H}}|, \qquad (|D| = \ell^a)$$

Turull Conjecture

$$|B_{a,m}| = |b_{a,m}|, \qquad (|D| = \ell^a)$$

Let k be a sufficiently large field of characteristic ℓ.

A *B-weight* of kG is a pair (P, M) where

- P is an ℓ-subgroup of G.
- M is a simple $kN_G(P)$-module in a block b of $N_G(P)$ inducing B.
- M is projective as a $kN_G(P)/P$-module.

G acts on B-weights by conjugation.

Alperin Weight Conjecture

$|\{(P, M)\}/\sim_G|$ is the number of simple kG-modules in B

Amidrunk Ordinary Conjecture

Suppose $O_\ell(G) = 1$ and B is a block of G with $D > 1$. Fix δ, r, $\mathcal{H}$ as before. Then

$$\sum_{\mathcal{C}/\sim_G,\, b} (-1)^{|\mathcal{C}|} |b_{\delta,r,\mathcal{H}}| = 0.$$

- $\mathcal{C}\colon 1 = E_0 < E_1 < \cdots < E_s$, a chain of elementary abelian ℓ-subgroups E_i.
- $/\sim_G$ means up to G-conjugacy, $|\mathcal{C}|$ is the length s of $\mathcal{C}$.
- b is a block of $N_G(\mathcal{C}) = \bigcap_i N_G(E_i)$ such that $b^G = B$.

Amidrunk Projective Conjecture

Suppose $Z \leq Z(G)$, $Z_\ell = O_\ell(G)$, and B is a block of G with $D > O_\ell(G)$. Fix $\zeta \in O_\ell(G)^\vee$ and fix δ, r, $\mathcal{H}$ as before. Then

$$\sum_{\mathcal{C}/\sim_G,\ b} (-1)^{|\mathcal{C}|} |b_{\delta,r,\mathcal{H},\zeta}| = 0.$$

- $\mathcal{C}:\ O_\ell(G) = E_0 < E_1 < \cdots < E_s$, a chain of ℓ-subgroups E_i such that all $E_i/O_\ell(G)$ is elementary abelian.
- $/\sim_G$ means up to G-conjugacy, $|\mathcal{C}|$ is the length s of $\mathcal{C}$.
- b is a block of $N_G(\mathcal{C}) = \bigcap_i N_G(E_i)$ such that $b^G = B$ and $b_{\delta,r,\mathcal{H},\zeta} = \{\psi \in b_{\delta,r,\mathcal{H}} : \psi \text{ covers } \zeta\}$

It has been said by E. T. Bell that “whenever groups disclosed themselves or could be introduced, simplicity crystallized out of comparative chaos”. This may often be true, but strangely enough, it does not apply to group theory itself, not even when we restrict ourselves to groups of finite order . . . we cannot answer some of the simplest questions. . . . This is why I am fascinated by the theory of finite groups.

Richard Brauer, Representations of Finite Groups

Finite reductive groups $\mathbf{G}^F$.

$\mathbf{G}$	connected reductive group over $\mathbf{F}_q$, $\mathbf{F} = \overline{\mathbf{F}}_q$
q	a power p^n of the prime p
F	Frobenius-like endomorphism defining $\mathbf{F}_q$-structure on $\mathbf{G}$
$\mathbf{T}$	F-stable torus, closed subgroup $\simeq \mathbf{F}^\times \times \mathbf{F}^\times \times \cdots \times \mathbf{F}^\times$,
	$P_{\mathbf{T}}(x) = \prod_d \phi_d(x)^{m_d}$, $\phi_d(x)$ cyclotomic, $\lvert \mathbf{T}^F \rvert = P_{\mathbf{T}}(q)$
$\mathbf{L}$	Levi subgroup, centralizer $\mathbf{C}_{\mathbf{G}}(\mathbf{T})$ of a torus $\mathbf{T}$,

$\mathbf{T}$ a maximal torus $\Rightarrow$ $\mathbf{T}$ a Levi subgroup

Example: $\mathbf{G} = \mathrm{GL}(n, \mathbf{F})$, $\mathbf{G}^F = \mathrm{GL}(n, \mathbf{F}_q)$, $F\colon (a_{ij}) \mapsto (a_{ij}^q)$.

An F-stable Levi $\mathbf{L}$ is a product of F-stable subgroups $\mathbf{L}_i$

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_0 & 0 & \cdots & 0 \\ 0 & \mathbf{L}_1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \mathbf{L}_t \end{pmatrix}$$

where $\mathbf{L}_i^F = \mathrm{GL}(n_i, \mathbf{F}_{q^{d_i}})$ and $n = \sum_{i=1} n_i d_i$. Then

$$\mathbf{Z}(\mathbf{L}) = \mathbf{Z}(\mathbf{L_0}) \times \mathbf{Z}(\mathbf{L_1}) \times \cdots \times \mathbf{Z}(\mathbf{L_t})$$

is a torus such that $\mathbf{L} = \mathbf{C_G}(\mathbf{Z}(\mathbf{L}))$ and $P_{\mathbf{Z}(\mathbf{L})}(x) = \prod_{i=0}^t (x^{d_i} - 1)$.

Suppose $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$.

- The Deligne-Lusztig linear operator $R_{\mathbf{L}}^{\mathbf{G}}$ takes characters of $\mathbf{L}^F$ into virtual characters of $\mathbf{G}^F$.
- Every χ in $(\mathbf{G}^F)^\vee$ is in $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for some $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus of $\mathbf{G}$ and $\theta \in (\mathbf{T}^F)^\vee$.
- Unipotent characters of $\mathbf{G}^F$ are the irreducible characters χ in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ as $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$.

Example: In $\mathrm{GL}(n, q)$ unipotent characters are the irreducible constituents in the action of $\mathrm{GL}(n, q)$ on maximal flags of the underlying vector space. They are labeled by partitions of n

Deligne-Lusztig: Geometric Conjugacy Classes

$$(\mathbf{G}^F)^\vee = \coprod_{(t)} \mathcal{E}(\mathbf{G}^F, (t))$$

- Write $(\mathbf{T}, \theta) \sim (\mathbf{T}', \theta')$ if $R_{\mathbf{T}}^{\mathbf{G}}(\theta)^\vee \cap R_{\mathbf{T}'}^{\mathbf{G}}(\theta')^\vee \neq \emptyset$.
- Let $\approx$ be the transitive extension of $\sim$.
- $\mathcal{E}(\mathbf{G}^F, (t)) = \bigcup_{(\mathbf{T},\theta)} R_{\mathbf{T}}^{\mathbf{G}}(\theta)^\vee$, where $(\mathbf{T}, \theta)$ in fixed $\approx$-equivalence class (t).

Such $\mathcal{E}(\mathbf{G}^F, (t))$ are labeled by semisimple dual conjugacy classes (t) of $\mathbf{G}^F$, that is, semisimple conjugacy classes (t) of $\mathbf{G}^{*F}$, where $\mathbf{G}^*$ is a related connected, reductive group called the dual of $\mathbf{G}$.

$\mathcal{E}(\mathbf{G}^F, (1))$ is the set of unipotent characters.

We suppose $\ell \neq p$ and consider ℓ-blocks B of $\mathbf{G}^F$.

A unipotent block B is one containing a unipotent character.

Set $\mathcal{E}(\mathbf{G}^F, B, (t)) = B \cap \mathcal{E}(\mathbf{G}^F, (t))$.

We call $\mathcal{E}(\mathbf{G}^F, B, (t))$ a Brauer-Lusztig block.

Broué-Michel

If B is a unipotent block, then

$$B = \coprod_{(t)} \mathcal{E}(\mathbf{G}^F, B, (t)),$$

where (t) runs over dual conjugacy classes of $\mathbf{G}^F$ of ℓ-th power order.

ℓ is excellent if

- ℓ is good in Steinberg-Springer sense.
- $\ell \neq 2$, $\ell \neq p$.
- ℓ does not divide $|\mathbf{Z}(\mathbf{G})^F/\mathbf{Z}^\circ(\mathbf{G})^F|$.
- ℓ does not divide $|\mathbf{Z}(\mathbf{G}^*)^F/\mathbf{Z}^\circ(\mathbf{G}^*)^F|$.
- $\mathbf{G}^F$ has no component of type ${}^3\mathbf{D}_4$ if $\ell = 3$.

If ℓ is excellent for $\mathbf{G}$, then ℓ is excellent for all F-stable Levi subgroups $\mathbf{M}$ of $\mathbf{G}$.

First consequence of excellent ℓ and (t) of ℓ-th power order:

Jordan Decomposition for $\mathcal{E}(\mathbf{G}^F, (t))$

There exist a pair $(\mathbf{G}(t), \hat{t})$, unique up to $\mathbf{G}^F$-conjugacy, where

- $\mathbf{G}(t)$ an F-stable Levi subgroup of $\mathbf{G}$,
- $\hat{t}$ a linear character of $\mathbf{G}(t)^F$ of order that of t,
- Deligne-Lusztig induction

$$R^{\mathbf{G}}_{\mathbf{G}(t)}(\hat{t}-)\colon \mathcal{E}(\mathbf{G}(t)^F, (1)) \to \mathcal{E}(\mathbf{G}^F, (t)), \quad \chi(t) \mapsto \pm\chi$$

is a bijection with signs. So $\chi = \pm R^{\mathbf{G}}_{\mathbf{G}(t)}(\hat{t}\chi(t))$.

Second consequence of ℓ excellent and t of ℓ-th power order:

- Let e be the multiplicative order of q modulo ℓ.

  $\mathbf{T}$ is a *ϕ_e-torus* if its order polynomial $P_{\mathbf{T}}(x) = \phi_e(x)^m$.

  $\mathbf{L}$ is an *e-split Levi* if $\mathbf{L} = \mathbf{C}_{\mathbf{G}}(\mathbf{T})$ for a ϕ_e-torus $\mathbf{T}$.

- Consider pairs $(\mathbf{M}, \mu)$, where $\mathbf{M}$ is e-split Levi, $\mu \in \mathcal{E}(\mathbf{M}^F, (s))$, and $(s) \subseteq (t)$. (The condition $(s) \subseteq (t)$ is well-defined since containment $\mathbf{M}^* \leq \mathbf{G}^*$ is unique up to $\mathbf{G}^{*F}$-conjugacy.)

- $(\mathbf{M}_1, \mu_1) \leq (\mathbf{M}_2, \mu_2)$ if $\mathbf{M}_1 \leq \mathbf{M}_2$ and $\langle \mu_2, R_{\mathbf{M}_1}^{\mathbf{M}_2}(\mu_1)\rangle_{\mathbf{M}_2^F} \neq 0$.

$(e,(t))$-Harish-Chandra Theory for $\mathcal{E}(\mathbf{G}^F,(t))$

- The relation $\leq$ is an ordering.
- The minimal $(\mathbf{M},\mu)$ in a fixed $(\mathbf{G},\chi)$ are $\mathbf{G}^F$-conjugate.
- $\mathcal{E}(\mathbf{G}^F,(t)) = \coprod_{(\mathbf{M},\mu)\ \min/\sim_{\mathbf{G}^F}} R_{\mathbf{M}}^{\mathbf{G}}(\mu)^\vee.$

(The case $(t)=(1)$ is due to Broué-Malle-Michel.)

Suppose $(\mathbf{M},\mu)$ minimal in $\mathcal{E}(\mathbf{G}^F,(t))$. By Jordan decomposition

$$\mu = \pm R_{\mathbf{M}(t)}^{\mathbf{M}}(\hat{t}\mu(t)).$$

A unipotent root of $(\mathbf{M},\mu)$ is a pair

$$(\mathbf{M},\mu)^\flat = ([\mathbf{M}(t),\mathbf{M}(t)], \mu(t)|_{[\mathbf{M}(t),\mathbf{M}(t)]^F}).$$

Example: $\mathbf{G} = \mathrm{GL}(n, \mathbf{F})$ and $\mathbf{G}^F = \mathrm{GL}(n, \mathbf{F}_q)$.

An e-split Levi subgroup $\mathbf{M}$ of $\mathbf{G} = \mathrm{GL}(n, \mathbf{F})$ has form

$$\mathbf{M}^F = \mathbf{M}_0^F \times \mathbf{M}_1^F \times \cdots \times \mathbf{M}_t^F,$$

where $\mathbf{M}_0^F = \mathrm{GL}(n_0, \mathbf{F}_q)$ and $\mathbf{M}_i^F = \mathrm{GL}(n_i, \mathbf{F}_{q^e})$ for $i > 0$.

Minimal pairs $(\mathbf{M}, \mu)$ associated to $\mathcal{E}(\mathbf{G}^F, (1))$ have form

- $\mathbf{M}_i^F = \mathrm{GL}(1, \mathbf{F}_{q^e})$ for $i > 0$.
- μ_0 is labeled by a partition of n_0 with no e-hooks.
- μ_i is the trivial character for $i > 0$.
- $(\mathbf{M}, \mu)^\flat = ([\mathbf{M}_0, \mathbf{M}_0], \mu_0|_{[\mathbf{M}_0, \mathbf{M}_0]^F})$.

$(e, (t))$-Harish-Chandra theory has everything to do with blocks!

Cabanes-Enguehard

- A Brauer-Lusztig block $\mathcal{E}(\mathbf{G}^F, B, (t))$ is an $(e, (t))$-Harish Chandra family $R_{\mathbf{M}}^{\mathbf{G}}(\mu)^{\vee}$.
- An $(e, (t))$-Harish Chandra family $R_{\mathbf{M}}^{\mathbf{G}}(\mu)^{\vee}$ is a Brauer-Lusztig block $\mathcal{E}(\mathbf{G}^F, B, (t))$.
- If $\mathcal{E}(\mathbf{G}^F, B, (t)) = R_{\mathbf{M}}^{\mathbf{G}}(\mu)^{\vee}$, $\mathcal{E}(\mathbf{G}^F, B', (t')) = R_{\mathbf{M}'}^{\mathbf{G}}(\mu')^{\vee}$, then
$$B = B' \iff (\mathbf{M}, \mu)^{\flat} \sim_{\mathbf{G}^F} (\mathbf{M}', \mu')^{\flat}.$$
- The defect groups of B are the Sylow ℓ-subgroups of $\mathbf{C}^{\circ}_{\mathbf{G}^F}([\mathbf{M}(t), \mathbf{M}(t)])$.

Composition of some bijections:

$$\begin{aligned} \mathcal{E}(\mathbf{G}^F, B, (1)) &\overset{\text{CE}}{=} R^{\mathbf{G}}_{\mathbf{M}}(\mu)^\vee \\ &\overset{\text{BMM}}{\simeq} (\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)/\mathbf{M}^F)^\vee \\ &\overset{\text{if } \mu \text{ extends}}{\simeq} (\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)|\mu)^\vee \end{aligned}$$

where $(\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)|\mu)^\vee = \{\chi \in \mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)^\vee \text{ covering } \mu\}$.

This bijection is in the [BMM] dictionary: Deligne-Lusztig induction of unipotent characters of e-split Levi subgroups corresponds to usual induction of characters in relative Weyl groups (by a Hecke algebra?).

This bijection is standard Clifford theory.

Generalizations to other Brauer-Lusztig blocks:

Let ρ be the composition of the following bijections:

$$
\begin{aligned}
\mathcal{E}(\mathbf{G}^F, B, (t)) &\overset{\text{CE}}{=} R_{\mathbf{M}}^{\mathbf{G}}(\mu)^\vee \\
&\overset{\text{Jordan twice}}{\simeq} R_{\mathbf{M}(t)}^{\mathbf{G}(t)}(\mu(t))^\vee \\
&\overset{\text{BMM}}{\simeq} (\mathbf{N}_{\mathbf{G}(t)^F}(\mathbf{M}(t), \mu(t))/\mathbf{M}(t)^F)^\vee \\
&\overset{\text{Frattini}}{\simeq} (\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)/\mathbf{M}^F)^\vee \\
&\overset{\text{if } \mu \text{ extends}}{\simeq} (\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)|\mu)^\vee
\end{aligned}
$$

where $(\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)|\mu)^\vee = \{\chi \in \mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)^\vee \text{ covering } \mu\}$.

A possible extension of $\rho\colon \mathcal{E}(\mathbf{G}^F, b, (t)) \to (\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)|\mu)^\vee$.

Equivariant Bijection Conjecture (EBC)

Suppose $\mathbf{G}$ is an e-split Levi of a connected, reductive group $\mathbf{H}$; $\tilde{\mathbf{G}}$ is a closed, F-stable subgroup of $\mathbf{H}$; and $\mathbf{G} \leq \tilde{\mathbf{G}} \leq \mathbf{N}_{\mathbf{H}}(\mathbf{G})$.

Then there exist an $\mathcal{G}_\ell$-equivariant signed bijection

$$\tilde{\rho}\colon \mathcal{E}(\tilde{\mathbf{G}}^F, b, (t)) \to (\mathbf{N}_{\tilde{\mathbf{G}}^F}(\mathbf{M}, \mu)|\mu)^\vee$$

such that $s(\tilde{\chi})/s(\tilde{\rho}(\tilde{\chi})) \equiv 1$ modulo ℓ for all $\tilde{\chi}$.

$$\mathcal{E}(\tilde{\mathbf{G}}^F, b, (t)) = \{\tilde{\chi} \in (\tilde{\mathbf{G}}^F)^\vee \text{ covering } \chi \text{ in } \mathcal{E}(\mathbf{G}^F, b, (t))\}$$
$$(\mathbf{N}_{\tilde{\mathbf{G}}^F}(\mathbf{M}, \mu)|\mu)^\vee = \{\tilde{\psi} \in \mathbf{N}_{\tilde{\mathbf{G}}^F}(\mathbf{M}, \mu)^\vee \text{ covering } \mu\}.$$

Ordinary Amidrunk Revisited

Suppose $H = \mathbf{H}^F$ is a finite reductive group with $O_\ell(H) = 1$.

Suppose B is an unipotent ℓ-block with defect group $D > 1$.

Suppose ℓ is excellent. Fix δ, r, $\mathcal{H}$ as before. Then

$$\sum_{\mathcal{C}/\sim_H,\, b} (-1)^{|\mathcal{C}|} |b_{\delta,r,\mathcal{H}}| = 0,$$

- $\mathcal{C} : 1 = E_0 < E_1 < \cdots < E_s$, a chain of elementary abelian ℓ-subgroups E_i.
- $/\sim_H$ means up to H-conjugacy, $|\mathcal{C}| = s$.
- b is a block of $N_H(\mathcal{C}) = \bigcap_i N_H(E_i)$ with $b^H = B$.

A possible proof:

- Need only good chains $\mathcal{C} : 1 = E_0 < E_1 < \cdots < E_s$

$$E_i = \Omega_1(O_\ell(\mathbf{Z}^\circ(\mathbf{C}^\circ_{\mathbf{H}}(E_i))^F) \text{ for all } i$$

For bad chains cancel each other in the sum.

- Centralizers $\mathbf{M}_i = \mathbf{C}^\circ_{\mathbf{H}}(E_i)$ of E_i in a good $\mathcal{C}$ form a chain

$$\mathcal{L}\colon \mathbf{H} = \mathbf{M}_0 > \mathbf{M}_1 > \cdots > \mathbf{M}_s$$

of e-split Levi subgroups. Moreover $\mathbf{N}_{\mathbf{H}^F}(\mathcal{C}) = \mathbf{N}_{\mathbf{H}^F}(\mathcal{L})$.

- Refine chain $\mathcal{L}$: $\mathbf{M}_0 > \mathbf{M}_1 > \cdots > \mathbf{M}_s$ by chains of Brauer-Lusztig triples:

$$\mathcal{A} : (\mathbf{M}_0, b_0, \mathcal{D}_0) > (\mathbf{M}_1, b_1, \mathcal{D}_1) > \cdots > (\mathbf{M}_s, b_s, \mathcal{D}_s).$$

Here b_i is a unipotent block of $\mathbf{M}_i^F$, $\mathcal{D}_i$ is a dual conjugacy class of $\mathbf{M}_i$, and containment $(\mathbf{M}, b, \mathcal{D}) \geq (\mathbf{M}', b', \mathcal{D}')$ means

$$\mathbf{M} \geq \mathbf{M}', \quad (b')^{\mathbf{M}^F} = b, \quad \mathcal{D} \supseteq \mathcal{D}'.$$

- A Brauer-Lusztig chain amounts to a chain $\mathcal{L}$ and a Brauer-Lusztig block $\mathcal{E}(\mathbf{M}_s^F, b, (t))$ of the last $\mathbf{M}_s$ of $\mathcal{L}$.

- $\mathbf{H}^F$-conjugate classes of Brauer-Lusztig chains come in pairs represented by $\mathcal{A}$ and $\mathcal{A}'$, where $\mathcal{A}'$ is $\mathcal{A}$ extended by a minimal Brauer-Lusztig triple.
- $\mathcal{A}$ and $\mathcal{A}'$ gives rise to the configuration of (EBC), namely $\mathcal{A}$ to $\mathcal{E}(\tilde{\mathbf{G}}^F, b, (t))$ and $\mathcal{A}'$ to $\mathbf{N}_{\tilde{\mathbf{G}}^F}(\mathbf{M}, \mu)|\mu)^\vee$.
- By (EBC) the terms in the alternating sum corresponding to $\mathcal{A}$ and $\mathcal{A}'$ cancel.

So all terms in the alternating sum cancel!

$\mathbf{H}$ is classical if $\mathbf{H}^F = \mathrm{GL}(V)$, $\mathrm{U}(V)$, $\mathrm{SO}(V)$, or $\mathrm{Sp}(V)$.

Broué-Fong-Srinivasan

(EBC) holds* for classical groups $\mathbf{H}$ if $\tilde{\mathbf{G}}$ is the normalizer of

$$\mathbf{H} = \mathbf{M}_0 > \mathbf{M}_1 > \cdots > \mathbf{M}_s = \mathbf{G},$$

a chain of e-split Levi subgroups of $\mathbf{H}$ ending in $\mathbf{G}$.

*Navarro property still open when $\mathbf{H}$ is $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$.

Such $\tilde{\mathbf{G}}$ suffice for the Amidrunk Conjecture.

On the good side: In classical groups the property

$$s(\tilde{\chi})/s(\tilde{\rho}(\tilde{\chi})) \equiv 1 \text{ modulo } \ell$$

rests on a corresponding property of the [BMM] bijection

$$\beta \colon R_{\mathbf{M}}^{\mathbf{G}}(\mu)^{\vee} \simeq (\mathbf{N}_{\mathbf{G}^F}(\mathbf{M}, \mu)|\mu)^{\vee},$$

when $(\mathbf{M}, \mu)$ is a minimal unipotent pair in $\mathbf{G}$ and μ extends:

$$s(\chi)/s(\beta(\chi)) \equiv 1 \text{ modulo } \phi_e(x)$$

The congruence modulo ℓ then holds because

- If $e|d$ and $\phi_{d\ell}(x)|s(\chi)$, then $\phi_d(x)|s(\chi)$.
- $|\mathbf{G}^F : \mathbf{L}^F|_p \equiv \pm 1$ modulo ℓ if $\mathbf{L}$ is an e-split Levi of $\mathbf{G}$.

On the bad side: The $\mathcal{G}_\ell$-equivariance of

$$\tilde{\rho}\colon \mathcal{E}(\tilde{\mathbf{G}}^F, b, (t)) \to (\mathbf{N}_{\tilde{\mathbf{G}}^F}(\mathbf{M}, \mu)|\mu)^\vee.$$

is open. The following case can occur in $\mathrm{SO}(V)$ and $\mathrm{Sp}(V)$:

$$\tilde{\rho}\colon \mathcal{E}(\mathrm{GL}(n,q)\langle\tau\rangle, (1)) \to (S(n) \times C_2)^\vee$$

- τ is the transpose-inverse operator on $\mathrm{GL}(n,q)$.
- τ fixes each unipotent character χ of $\mathrm{GL}(n,q)$.
- $\mathcal{E}(\mathrm{GL}(n,q)\langle\tau\rangle, (1)) = \{\text{extensions } \tilde{\chi} \text{ of such } \chi\}$.

Gow has conjectured $\mathbf{Q}(\tilde{\chi}) \subseteq \mathbf{Q}(\sqrt{q})$. If $q > n$, this is a corollary of recent work of Waldspurger, and then $\tilde{\rho}$ is $\mathcal{G}_\ell$-equivariant.

Open: The analogous case in $\mathrm{SO}(V)$ and $\mathrm{Sp}(V)$.

$$\tilde{\rho}\colon \mathcal{E}(\mathrm{U}(n,q)\langle\tau\rangle,(1)) \to (S(n)\times C_2)^{\vee},$$

I am convinced that the special problems in all their complexity constitute the stock and core of mathematics ... The general theories are shown here as springing forth from special problems whose analysis leads to them with almost inevitable necessity as the fitting tools for their solution. Once developed, these theories spread their light over a wide region beyond their limited origins.

Hermann Weyl, The Classical Groups