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Canonical bases and construction of Chevalley groups over $\mathbb{Z}$.

Root system

$$R = (Y, X, \underset{\substack{\cap \\ Y}}{\alpha_i^\vee}, \underset{\substack{\cap \\ X}}{\alpha_i} \ (i \in I), \langle\ \rangle)$$

Lie theory: $R \Rightarrow 4$ important objects

Reductive group $G_{\mathbb{C}}$	Lie Alg $\mathfrak{g}_{\mathbb{C}}$	Envel alg $U_{\mathbb{C}}$	coord. ring $\mathcal{O}_{\mathbb{C}}$
[Chevalley 1955 $G_{\mathbb{C}}$-adjoint] $G_A \subset \mathrm{Aut}(\mathfrak{g}_A)$	$\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_A = \mathfrak{g}_{\mathbb{Z}} \otimes A$		A field, V faithful $G_{\mathbb{C}}$-mod
[Chevalley 1960-61 $G_{\mathbb{C}}$: semis] $G_A \subset \mathrm{Aut}(V_A)$			$\mathcal{O}_{\mathbb{Z}}$, $\mathcal{O}_A = A \otimes \mathcal{O}_{\mathbb{Z}}$; $V_{\mathbb{Z}} = \mathbb{Z}$ form, $V_A = A \otimes V_{\mathbb{Z}}$
Demazure Grothendieck 1963			uniqueness for $\mathcal{O}_{\mathbb{Z}}$

Kostant 1966 ($G_{\mathbb{C}}$ simply connected)

$U_{\mathbb{Z}} \subset U_{\mathbb{C}}$ $\qquad$ $\mathcal{O}_{\mathbb{C}} = \mathrm{Hom}_{res}(U_{\mathbb{C}}, \mathbb{C})$

$U_A = U_{\mathbb{Z}} \otimes A$ $\qquad$ $\mathcal{O}_{\mathbb{Z}} = \{ f \in \mathcal{O}_{\mathbb{C}} ; f(U_{\mathbb{Z}}) \subset \mathbb{Z} \}$.

$$\mathcal{O}_A = A \otimes \mathcal{O}_{\mathbb{Z}}$$

$G_A = \{ \varphi : \mathcal{O}_A \to A ;$ A-alg. hom preserving 1 $\}$.

states w/o proof:

A - alg. closed field $\Rightarrow$ $\mathcal{O}_A$ is a fin. gen integral domain coord ring of a conn. red grp/A of type R

This is now a theorem

U_φ : assoc. alg. w/ 1 generated by
$e_i, f_i, y \in Y$, relations
$i \in I$

standard
relations

write formally $Y = \sum_{x \in X} \langle y, \lambda \rangle 1_\lambda$. Then $1_\lambda 1_{\lambda'} = \delta_{\lambda\lambda'} 1_\lambda$

$U_\varphi \subset U' \supset \dot{U}_\varphi$

U': assoc. alg w/ 1 generated by.
$e_i, f_i \quad \sum_{\lambda \in X} a_\lambda 1_\lambda$ (formal sum, $a_\lambda \in \varphi$)
$i \in I$
relations

$1_\lambda e_i = e_i 1_{\lambda + \alpha_i}$

$\vdots$

$\dot{U}_{Q}$ ideal of U' generated by 1_λ ($\lambda \in X$).

As a ring generated by 1_λ, $e_i 1_\lambda$, $f_i 1_\lambda$,

has no 1

$\underbrace{\text{First appeared}}_{\text{Type } A}$ in Beilinson - Deligne - Macpherson Duke MJ 1990

$\dot{U}_{\mathbb{Z}}$ subring of $\dot{U}_{Q}$ gen'd by

$$\frac{e_i^n}{n!} 1_\lambda, \quad \frac{f_i^n}{n!} 1_\lambda, \quad n \geq 0$$

<u>claim</u>:

$\dot{U}_{\mathbb{Z}}$ has <u>canonical</u> $\mathbb{Z}$ <u>basis</u> $\dot{B}$

which is also Q-basis of $\dot{U}_{Q}$

(Lusztig, Proc. NAS 1992).

<u>Example</u>

$$\frac{e_i^n}{n!} 1_\lambda \in \dot{B} \qquad \frac{f_i^n}{n!} 1_\lambda \in \dot{B}$$

$$\frac{f_i^a}{a!} 1_\lambda \frac{e_i^b}{b!} \in \dot{B} \qquad \langle \alpha_i^\vee, \lambda \rangle = a+b.$$

$\dot{B}$ is compatible w/ various filtrations

eg. let $\lambda \in X^+$ (dominant weight $\langle \alpha_i^\vee, \lambda \rangle \in \mathbb{N}$)

V_λ : $\dot{U}_Q$-mod with highest wt λ.

$\dot{U}^{\geq \lambda} = \{ u \in \dot{U} ;\ u/_{V_{\lambda'}} \neq 0 \Rightarrow \lambda' \geq \lambda \}$: two sided ideal.

Then $\dot{U}^{\geq \lambda} \cap \dot{B}$ is a basis of $\dot{U}^{\geq \lambda}$.

f - Q alg w/ 1 gens. θ_i $i \in I$
+ serre relations

$f \rightrightarrows U'$ $\qquad \theta_i \to \theta_i^+ = e_i$

$\qquad \theta_i \to \theta_i^- = f_i$

f has a canonical basis B

$\dot{B}^+ := \{ b^+ 1_\lambda ;\ \begin{smallmatrix} b \in B \\ \lambda \in X \end{smallmatrix} \} \subset \dot{B}$

$\dot{B}^- := \{ b^- 1_\lambda ;\ \begin{smallmatrix} b \in B \\ \lambda \in X \end{smallmatrix} \} \subset \dot{B}$

A comm. ring w/ 1.

$\dot{U}_A = A \otimes \dot{U}_{\mathbb{Z}}$ $\qquad$ A - alg. w/o 1, canonical basis $1 \otimes \dot{B}$

$\dot{U}_A^{\pm} = A$ - submod of U_A spanned by $\dot{B}^{\pm}$

Def $\mathcal{O}_{\mathbb{Q}} = \mathrm{Hom}_{\mathrm{res}}(\dot{U}_{\mathbb{Q}}, \mathbb{Q})$

claim $\left\{ b^*: \dot{U}_{\mathbb{Q}} \to \mathbb{Q} \;\middle|\; \begin{array}{l} \mathbb{Q}\text{-linear} \\ b^*(b') = \delta_{bb'} \end{array} \right\}$ is a $\mathbb{Q}$-basis of $\dot{U}_{\mathbb{Q}}$

$b, b' \in \dot{B}$

$\mathcal{O}_{\mathbb{Z}} = \{ f \in \mathcal{O}_{\mathbb{Q}} \mid f(\dot{U}_{\mathbb{Z}}) \subset \mathbb{Z} \}$ and a $\mathbb{Z}$ basis of $\mathcal{O}_{\mathbb{Z}}$

$\mathcal{O}_A = A \otimes \mathcal{O}_{\mathbb{Z}}$ A-alg

claim: $\mathcal{O}_{\mathbb{Q}}, \mathcal{O}_{\mathbb{Z}}, \mathcal{O}_A$ are the same as Kostants

If $a \in \dot{B}^{\pm}$ then $a^*: \dot{U}_A \to A$ restricts to a linear form on $\dot{U}_A^{\pm}$ denoted again by a.

$\mathcal{O}_A^{\pm}$: A-submod of $\mathrm{Hom}_A(\dot{U}_A^{\pm}, A)$ spanned by a^* $(a \in \dot{B}^{\pm})$.

$\mathcal{O}_A^{\pm}, \mathcal{O}_A$ ~~is~~ are naturally ~~a~~ Hopf algebras

$\pi^{\pm}: \mathcal{O}_A \to \mathcal{O}_A^{\pm}$ A-linear map

$a^* \to \begin{cases} a^*, & a \in \dot{B}^{\pm} \\ 0, & a \in \dot{B} - \dot{B}^{\pm} \end{cases}$

is a surj. alg hom.

<u>define</u>

$$\begin{tikzcd} \iota : \mathcal{O}_A \arrow[rr] \arrow[dr, "\text{co-mult}"'] & & \mathcal{O}_A^- \otimes_A \mathcal{O}_A^+ \\ & \mathcal{O}_A \otimes_A \mathcal{O}_A \arrow[ur, "\pi^- \otimes \pi^+"'] & \end{tikzcd}$$

<u>prop</u> ι is a split injection

Idea of proof $\mathcal{O}_A$ has basis $\dot{B}$

$\mathcal{O}_A^- \otimes \mathcal{O}_A^+$ " " $\dot{B}^- \times \dot{B}^+$

for any $\lambda \in X^+$ (dominant wt)

let
$$Z_\lambda = \left\{ (a,b) \in \dot{B}^- \times \dot{B}^+ \;\middle|\; a, b \in \dot{B}[\lambda],\ \begin{matrix} a 1_\lambda = a \\ 1_\lambda b = b \end{matrix} \right\}.$$

$Z_\lambda \xrightarrow{\sim} \dot{B}[\lambda]$ $\quad (a,b) \to c$

where $ab = c +$ terms in higher $B[\lambda]$.

$$Z = \bigsqcup_{\lambda \in X^+} Z_\lambda \subset \dot{B}^- \times \dot{B}^+ \subset \mathcal{O}_A^- \otimes \mathcal{O}_A^+$$

define $u : \mathcal{O}_A^- \otimes \mathcal{O}_A^+ \longrightarrow \{A\text{-span of } Z\}$

$$a^* \otimes b^* \longrightarrow \begin{cases} a^* \otimes b^* & \text{if } (a,b) \in Z \\ 0 & \text{" } (a,b) \notin Z \end{cases}$$

Enough to show: $\alpha \cdot \iota$ = isomorphism

since $Z \longleftrightarrow \dot{B}$ $\qquad (a, b) \longleftrightarrow c$

$ab = c +$ higher

this is a lin. map between modules given by an upper triang-ular matrix w/ respect to $\dot{B}$

Note $\mathcal{O}_A^{+} \cong A[t_1, t_2, \ldots, t_m] \otimes A[x]$

as A-algebra.

Hence. $\mathcal{O}_A$ is a subalgebra of

$A[t_1, \ldots, t_m] \otimes A[t_1', \ldots, t_m'] \otimes A[x] \otimes A[x]$

so $\mathcal{O}_A$ is integral domain whenever

A " " "

Let $\hat{U}_A$ = set of formal A-linear combinations of element $b \in \dot{\mathcal{B}}$

it contains $\dot{U}_A$. The A-algebra structure of $\dot{U}_A$ extends to an A-algebra structure of $\hat{U}_A$ (has 1).

We can view $\hat{U}_A = \mathrm{Hom}_A(\mathcal{O}_A, A)$.

Hence $G_A = \mathrm{Hom}_{A\text{-alg}}(\mathcal{O}_A, A)$ is a subset of $\hat{U}_A$.

In fact it is a subgrp. of group of units of $\hat{U}_A$.

Hence each element of G_A can be expanded as $\sum_b c_b b$, $c_b \in A$

<u>Example</u>

$i \in I$ $h \in A$

$$x_i(h) = \sum_{\substack{c \in \mathbb{N} \\ \lambda \in X}} h^c \overbrace{\left(\frac{e_i^c}{c!} 1_\lambda\right)}^{\dot{\mathcal{B}} \ni} \in G_A$$

$$y_i(h) = \sum_{\substack{c \in \mathbb{N} \\ \lambda \in X}} h^c \underbrace{\left(\frac{f_i^c}{c!} 1_\lambda\right)}_{\in \dot{\mathcal{B}}}$$

$v \in A^{\text{units}}$ $\qquad t_i(v) = \sum_{\lambda \in X} v^{\langle \alpha_i^\vee, \lambda \rangle} 1_\lambda \in G/_A$

$i \in I$

$$s_i' = \sum_{a+b=\langle \alpha_i^\vee, \lambda \rangle} (-1)^b \underbrace{\frac{f_i^a}{a!} 1_\lambda \frac{e_i^b}{b!}}_{\in \dot{B}} \in G/_A$$

s_i' satisfies braid relations.