Superbosonization of invariant matrix ensembles

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joint work with
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The simplest case

Let $G = GL_n(\mathbb{C})$, or one of the other classical groups Sp_n, O_n, set

$$V = \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) = M_{n,q}(\mathbb{C}) \oplus M_{q,n}(\mathbb{C})$$

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$\mathcal{A}_V$ comes equipped with natural grading. Given $f \in \mathcal{A}_V^G$, we are interested in an "explicit" formula for the projection

$$\Omega_V : \mathcal{A}_V^G \to \Lambda^{2qn} V^*, \quad f \mapsto \Omega_V(f) \quad \Big(\text{sometimes } \int f\Big)$$

the map is called the *Berezin integral* of f.

Interested in asymptotic behaviour for $n \to \infty$.

Superbosonization - A brief characterization

Given ensemble of disordered Hamiltonians (say hermitian $n \times n$ matrices), probabilty distribution, rapid decay at infinity, or bounded support.
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Supersymmetry method: idea is to transfer expectation values w.r.t. the probability measure $d\mu(H)$ to certain (super)integrals.

Starting point: characteristic function of the probabilty measure of a given ensemble of disordered Hamiltonians.

$$\mathcal{F}(K) = \int e^{-i\mathrm{Tr}(KH)} d\mu(H)$$

What is K?

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Characteristic function is evaluted on a supermatrix K, entries are commuting and anti-commuting variables (which depend on what observables are to be computed).

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In our case

$$V = V_0 \oplus V_1, \begin{cases} V_0 = M_{n,p} \oplus M_{p,n} \text{ commuting variables } z_{i,j}, \tilde{z}_{j,i} \\ V_1 = M_{n,q} \oplus M_{q,n} \text{ anti-commuting variables } \zeta_{k,l}, \tilde{\zeta}_{l,k} \end{cases}$$

p-bosonic and q-fermionic copies (we will see later why the $\tilde{}$-variables come into the picture).

Superbosonization - A brief characterization

For such a situation, in general the matrix entries of K will be of the form

$$K_{i,j} = \sum_{l=1}^{p} z_{i,l}\tilde{z}_{l,j} + \sum_{m=1}^{q} \zeta_{i,m}\tilde{\zeta}_{m,j}$$

where $z_{i,j}, \tilde{z}_{j,i}$ commuting variables, $1 \leq i \leq n, 1 \leq j \leq p$
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To calculate the spectral correlation function for example, one multiplies the characteristic function by the exponential function

$$f = exp\left(i \sum_{l,l'} z_{l,l'} E_{l'} \tilde{z}_{l',l} + i \sum_{k,k'} \zeta_{k,k'} F_{k'} \tilde{\zeta}_{k',k} \right) \mathcal{F}(K)$$

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Note: f is a map $f : V_0 \to \Lambda^\bullet V_1^*$, a so-called *superfunction*.
parameters E_l, F_l - physical meaning of energy

Superbosonization - A brief characterization

Leaving out many details, to calculate for example the spectral correlation function one ends up with the following problem: have a holomorphic map or superfunction

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$$\Omega_V(f) = \int_{M_{n,p}} D_{z,\tilde{z},\zeta,\tilde{\zeta}} f(z,\tilde{z},\zeta,\tilde{\zeta})$$

where $M_{n,p}$ is embedded in $V_0 = M_{n,p} \oplus M_{p,n}$ as a real subspace via

$$A \mapsto (A, \overline{A}^t).$$

Assume: f analytic on the *diagonal* $M_{p,n}$, rapid decay.
Remark: doubling of the variables, complexification of a real situation

Superbosonization - A brief characterization

For the formula

$$\Omega_V(f) = \int_{M_{n,p}} D_{z,\tilde{z},\zeta,\tilde{\zeta}} f(z, \tilde{z}, \zeta, \tilde{\zeta})$$

remains to explane the Berezin integral form

$$D_{z,\tilde{z},\zeta,\tilde{\zeta}} = \prod_{a=1}^{p} \prod_{i=1}^{n} dz_{i,a} d\tilde{z}_{a,i} \prod_{b=1}^{q} \prod_{j=1}^{n} \frac{\partial^2}{\partial \zeta_{j,b}\, \partial \tilde{\zeta}_{b,j}} \,.$$

First factor: Lebesgue measure, second factor: projection onto the highest degree component of f.

Summarizing:

$V = V_0 \oplus V_1$ where

- $V_0 = M_{n,p} \oplus M_{p,n}$ commuting variables $z_{i,j}, \tilde{z}_{i,j}$
- $V_1 = M_{n,q} \oplus M_{q,n}$ anti-commuting variables $\zeta_{i,j}, \tilde{\zeta}_{i,j}$
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- $G = GL_n(\mathbb{C})$ (or a classical group: $Sp_{2m}(\mathbb{C}), O_n(\mathbb{C})$)
- everything is G-invariant (probability measure) respectively equivariant
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Superbosonization (Efetov): use the additional symmetry to simplify the Berezin integral
- assume $n \geq p$ for $G = GL_n(\mathbb{C})$ ($n \geq 2p$ for the other classical groups)

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$$V = V_0 = M_{n,p} \oplus M_{p,n},$$

$f : V_0 \to \mathbb{C}$ is just a G-invariant holomorphic function.

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Classical invariant theory: (recall $n \geq p$)
GL_n-invariant polynomials on V "are the same" as polynomials on $M_{p,p}$ via the map

$$P \in \mathbb{C}[M_{p,p}] \to p \in \mathbb{C}[V] \text{ defined by } p(A, B) := P(BA)$$

Same holds for holomorphic invariant functions (Luna).

Every holomorphic invariant function on V is the pull back of the form $f(A, B) = F(BA)$, F a holomorphic function on $M_{p,p}$.

The case $q = 0$, $G = GL_n$

Theorem. f invariant holomorphic function on V, F the corresponding holomorphic function on $M_{p,p}$, then the Berezin integral

$$\Omega_V(f) = factor \int_{D_p = p\times p \ \ pos.herm.matrices} F(x)\det^n(x)\,d\mu_{D_p}.$$

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Let $\pi : V_0 = M_{n,p} \oplus M_{p,n} \to M_{p,p}$, $(A, B) \mapsto BA$.
Then the image of the "diagonally" embedded matrices

$$M_{n,p} \hookrightarrow M_{n,p} \oplus M_{p,n}, \quad A \mapsto (A, \overline{A}^t)$$

with respect to π are the $p \times p$-matrices of the form $\overline{A}^t A$ for some $A \in M_{n,p}$, so the non-negative hermitian $p \times p$-matrices.

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Theorem: *Suppose $n > q$. For $f \in \Lambda^\bullet(V_1^*)^{GL_n}$ there exists a function $F \in \mathbb{C}[M_{q,q}(\mathbb{C})]$ and an appropriate Haar measure on U_q such that*

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Remark: Similar statement for the other classical groups, get integrals over compact symmetric spaces.

Superbosonization formula:

$$G = GL_n(\mathbb{C})$$

$$V_0 = M_{n,p} \oplus M_{p,n}, \quad V_1 = M_{n,q} \oplus M_{q,n}$$

$$W_0 = M_{p,p} \oplus M_{q,q}, \quad W_1 = M_{p,q} \oplus M_{q,p}.$$

Lemma: If $n > p, q$, then $\exists$ surjective homomorphism between
- holomorphic maps $F : W_0 \to \Lambda^\bullet(W_1^*)$

and
- holomorphic $G = GL_n$-equivariant maps $f : V_0 \to \Lambda^\bullet(V_1^*)$.

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$(S^\bullet V_0^* \otimes \Lambda V_1^*)^G =$ (invariant) graded symmetric algebra

$$= \left(T(V_0^* \oplus V_1^*) / < x \otimes x' + (-1)^{|x||x'|} x' \otimes x > \right)^G$$

Superbosonization formula:

Now the degree 2 part is

$$\begin{aligned} S^2(V_0^* \oplus V_1^*)^G &= S^2(V_0^*)^G \oplus (\Lambda^2 V_1^*)^G \bigoplus (V_0^* \otimes V_1^*)^G \\ &= M_{p,p}^* \oplus M_{q,q} \bigoplus M_{p.q} \oplus M_{q,p} \\ &= W_0^* \bigoplus W_1^* \end{aligned}$$

Isomorphism of vector spaces descends to a surjective map (Howe)

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Schwarz: hol. equiv. maps $f = f_1 f_2$, f_2 algebraic equiv. map, f_1 hol. invariant function on V_0.

Superbosonization formula:

$G = GL_n(\mathbb{C})$

Theorem:(Superbosonization formula) Let $F : W_0 \to \Lambda^\bullet(W_1^*)$ be a lift for $f : V_0 \to \Lambda^\bullet(V_1^*)$, then the Berezin integral:

$$\Omega_V(f) = \text{factor} \int_D proj_{high.\ deg.} J(x,y) SDet^n(x,y) F(x,y) d\mu_{Dp} dk$$

where where

$$D = (\textit{space of positive hermitian } p \times p \textit{ matrices}) \times U_q$$

and $SDet$ = superdeterminant / Berezinian

$$SDet \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = \frac{\det(x)}{\det(y - \tau x^{-1}\sigma)}$$

and $J(x,y) = \det^q(x)\det^q(y)/\det^{q-p}(y - \tau x^{-1}\sigma)$.

Superbosonization formula:

The general case: $G = O_n, Sp_n$

$V_0 = M_{n,p} \oplus M_{p,n}, \quad V_1 = M_{n,q} \oplus M_{q,n}$

$W_0 = Sym_{2p,2p} \oplus Alt_{2q,2q}, \quad W_1 = M_{2p,2q}$ for $G = O_n$.

$W_0 = Alt_{2p,2p} \oplus Sym_{2q,2q}, \quad W_1 = M_{2p,2q}$ for $G = Sp_n$.

Lemma: If $\geq 2p$, then $\exists$ surjective homomorphism between
- holomorphic maps $F : W_0 \to \Lambda^\bullet(W_1^*)$
and
- holomorphic G-equivariant map $f : V_0 \to \Lambda^\bullet(V_1^*)$.

Set $J(x, y) = \det^q(x) \det^{q-m/2}(y) / \det^{q-m/2-p}(y - \tau x^{-1}\sigma)$.

Here $m = 1, -1$ for $G = O_n, Sp_n$.

Superbosonization formula:

The general case: $G = GL_n, O_n, Sp_n$

Theorem: (Superbosonization formula)
Let $F : W_0 \to \Lambda^\bullet(W_1^*)$ be a lift for the G-equivariant holomorphic map $f : V_0 \to \Lambda^\bullet(V_1^*)$, then the Berezin integral:

$$\Omega_V(f) = \text{factor} \int_D proj_{high.\ deg}\, J(x,y) SDet^{n'}(x,y) F(x,y) d\mu_{Dp} dk$$

where $n' = n/(1+|m|) \geq p$, $m = 0, 1, -1$ for $G = GL_n, O_n, Sp_n$, and the domain for the integration is:

$D = GL_p(\mathbb{C})/U_p \times U_q$ for $G = GL_n$

$D = GL_{2p}(\mathbb{R})/O_{2p} \times U_{2q}/USp_{2q}$ for $G = O_n$

$D = GL_p(\mathbb{H})/USp_{2p} \times U_{2q}/O_{2q}$ for $G = Sp_n$

Two remarks:

- factors can be made precise, so formulas can be used for calculations

- method extends to case where one uses products of these groups.

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1st step Howe duality: For simplicity n even, $G = GL_n$.

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Set $N = qn$, let $\mathcal{C}(V \oplus V^*)$ be the Clifford algebra:

$$\mathcal{C}(V \oplus V^*) = \langle V \oplus V^* \oplus \mathbb{C} \rangle / ww' + w'w = s(w, w') \cdot 1,$$

where for

$$w = v + \phi, w' = v' + \phi' \in V \oplus V^* : \ s(v + \phi, v' + \phi') = \phi'(v) + \phi(v').$$

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Note: linear span of $ww' - w'w$ stable under $[\cdot, \cdot]$
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Set $G' =$ centralizer in $Spin_{4N}$ of G, then $G' = GL_{2q}$.

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get *Spin*-representation of $Spin_{4N}$ on $\Lambda^\bullet V^*$

action of G on V, V^* gives rise to a map $\phi : GL_n \to Spin_{4N}$

Set G' = centralizer in $Spin_{4N}$ of G, then $G' = GL_{2q}$.

To see this, recall: $V \oplus V^* \simeq \mathbb{C}^n \otimes \mathbb{C}^{2q*} \oplus \mathbb{C}^{n*} \otimes \mathbb{C}^{2q}$.

The proof for $p = 0$:

By Howe duality:

$\Lambda^\bullet V^*$ ist a direct sum

$$\Lambda^\bullet V^* \simeq \bigoplus U_i \otimes N_i$$

where U_i, N_i irreducible G resp. G'-modules, and $\forall i \neq j$: $U_i \not\simeq U_j, N_i \not\simeq N_j$.

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In particular: $\mathcal{A}_V^G = \Lambda^\bullet(V^*)^G$ is an irreducible G'-module:

$$\mathcal{A}_V^G \simeq V(\frac{n}{2}, \ldots, \frac{n}{2}, -\frac{n}{2}, \ldots, -\frac{n}{2}).$$

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Note: $\Lambda^0(V^*) \subset \Lambda^\bullet(V^*)^G$ and $\Lambda^{2N}(V^*) \subset \Lambda^\bullet(V^*)^G$ are lowest / highest weight vectors. So Berezin integral becomes projection onto the lowest weight vector.

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Interpretation as functions:

$$\mathfrak{g}' = \mathrm{Lie}\, GL_{2q}, \quad \mathfrak{g}' = \mathfrak{u}^- \oplus \mathfrak{h} \oplus \mathfrak{u}^+$$

where

$$\mathfrak{u}^- = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \mathfrak{h} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \mathfrak{u}^+ = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix},$$

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parabolic Verma module

$$M(\lambda) = U(\mathfrak{g}') \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}^+)} \mathbb{C}_\lambda$$

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and hence

$$\mathrm{Hom}_H(V(\lambda), V(\lambda)_{-\lambda}) = \mathrm{Hom}_H(M(\lambda), V(\lambda)_{-\lambda})$$

is one dimensional and spanned by the Berezin integral

The proof for $p = 0$:

Now

$$\mathrm{Hom}_H(M(\lambda), V(\lambda)_{-\lambda}) = \mathrm{Hom}_H(\mathbb{C}[M_{q,q}] \otimes \mathbb{C}_{2\lambda}, \mathbb{C})$$

and, as H-modules ($H = GL_q \times GL_q$):

$$\mathbb{C}[GL_q] = \mathbb{C}[M_{q,q}]_{\det} \supset \mathbb{C}[M_{q,q}] \otimes \mathbb{C}_{2\lambda} = \frac{1}{\det^n}\mathbb{C}[M_{q,q}]$$

Here one has an obvious projector onto the invariants

$$\mathbb{C}[GL_q] \to \mathbb{C}, \quad f \mapsto \int_{U_q} f dk$$

It follows:

$$f \in \Lambda^\bullet V^G \to u \otimes z_\lambda \in U(\mathfrak{u}^-) \otimes \mathbb{C}_\lambda \to F(x)\det{}^{-n}(x) \in \mathbb{C}[M_{q,q}] \otimes \mathbb{C}_{2\lambda}$$

and for the Berezin integral: $\Omega_V(f) = \int_{U_q} F(k)\det{}^{-n}(k)\, dk$.

Summarizing:

- Advantage of the new method: by conversion from its original role as the number of integrations to do, the (usually) big integer n has been turned into an exponent
- applicable in cases where other methods (non-Gaussian distribution) did not work so far
- even in cases where other methods work get interesting equalities. As an example, Martin Zirnbauer applied the new method Wegners n-orbital model with n orbitals per site and unitary symmetry. Be warned, however, that this equivalence of the formulas obtained by the new method and Hubbard-Stratonovich is by no means easy to see directly.