Coxeter Group Actions on the Cohomology of Toric Varieties

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Introduction

A toric variety is generally a torus, plus some “boundary components”.

There is one associated with every crystallographic root system \( \Phi \), which naturally has a \( W \)-action, \( W \) being the associated Weyl group.

The action of \( W \) on \( H^*(\mathcal{I}_W(\mathbb{C})) \) is a classical subject, which has been discussed by Procesi, Dolgachev, Lunts, Stembridge and others.

In general, if a variety \( X \) is defined over a number field \( K \), it has an associated \( \mathbb{C} \)-variety \( X(\mathbb{C}) \), but may also be reduced mod a prime ideal \( p \), to give an \( \overline{\mathbb{F}}_q \)-variety \( X_p \).
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The method of counting fixed points of twisted Frobenius maps on $X_p$ is sometimes very effective in computing group actions.

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The basic idea

Let $X$ be a variety over $K \subset \mathbb{C}$, a number field, and let $G$ be a finite group of $K$-morphisms of $X$.

Problem: describe the (graded) action of $G$ on the usual (Betti, or singular) cohomology $H^*(X, \mathbb{C})$.

Interpret as: compute for any $g \in G$

$$P_X(g, t) := \sum_i \text{trace}(g, H^i(X, \mathbb{C})) t^i.$$
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The residue field $k(p)$ of $p$ has order (say) $q$, and we write $X_q$ for the $\overline{\mathbb{F}_q}$-variety associated with $X$.

There are two elements to the method: first, given an isomorphism $\overline{\mathbb{Q}_\ell} \sim \mathbb{C}$, we have isomorphisms of $G$-modules

$$H^i(X(\mathbb{C}), \mathbb{C}) \sim H^i(X_q, \overline{\mathbb{Q}_\ell}).$$

In practice, one has such results for “almost all $q$”, and we shall assume this.

Second, assume that we knew that the Frobenius morphism $\mathcal{F}$ acts on $H^i_c(X_q, \overline{\mathbb{Q}_\ell})$ with just a single eigenvalue $q^{m_i}$. 
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For any $g \in G$, compute $|X_q^{g,F}|$ using Grothendieck’s fixed point theorem:

$$|X_q^{g,F}| = \sum_i (-1)^i \text{trace}(gF, H^i_c(X_q, \overline{\mathbb{Q}_\ell}))$$

$$= \sum_i (-1)^i q^{m_i} \text{trace}(g, H^i_c(X(\mathbb{C}), \mathbb{C})).$$

If we know the left side for almost all $q$, and $i \mapsto m_i$ is injective, then we have the compact supports version of $P_X(g, t)$.

In many examples (e.g. for $X$ smooth) the compact supports version is enough.
For any \( g \in G \), compute \( |X^g| \) using Grothendieck’s fixed point theorem:

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|X^g| = \sum_i (-1)^i \text{trace}(g, H^i_c(X, \overline{Q}_\ell)) = \sum_i (-1)^i q^m \text{trace}(g, H^i_c(X(\mathbb{C}), \mathbb{C})).
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Cohomology and Filtrations.
The setup.

\(K\): an algebraic number field, \(\overline{K}\): its algebraic closure.

\(S\): a finite set of primes of \(K\).

\(K_S \subset \overline{K}\): the maximal subfield of \(\overline{K}\), unramified outside \(S\).

\(G := \text{Gal}(\overline{K}/K) \xrightarrow{\text{onto}} G_{K,S} := \text{Gal}(K_S/K)\).

These are both profinite topological groups; subgroups of finite index are open.

\(\ell\): a rational prime, all of whose prime factors in \(K\) lie in \(S\).

If \(p \not\in S\) is a prime of \(K\), there is an element \(\text{Frob}_p \in G_{K,S}\) well defined up to conjugation.

If \(q_p := |\kappa(p)|\) (\(\kappa(p)\) is the residue field of \(p\)) then \(\text{Frob}_p\) induces the \(q_p\)-power map on the extension of \(\kappa(p)\) arising in \(K_S\).
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Cohomology theories

Let $X$ be an algebraic variety (i.e. a reduced scheme of finite type) over the number field $K$.

There are 3 cohomology theories naturally associated with $X$. The interrelationships among them are the key to this work.

1. de Rham Cohomology. This is a sequence $H^j_{dR}(X)$, $j = 0, 1, 2, \ldots$ of $K$-vector spaces, which come naturally with a filtration $F^\bullet H^j_{dR}(X)$:

$$F^k H^j_{dR}(X) \supseteq F^{k+1} H^j_{dR}(X).$$

2. Betti (usual) Cohomology. For any embedding $\sigma: K \hookrightarrow \mathbb{C}$, $X_{\sigma} := X \otimes_K \mathbb{C}$ has $\mathbb{C}$-points which may be identified with a complex analytic (algebraic) variety $X_{\sigma}(\mathbb{C})$. 
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Its complex cohomology $H^i(X_{\sigma}(\mathbb{C}), \mathbb{C})$ is a sequence of $\mathbb{C}$-vector spaces.

Betti cohomology comes with 2 natural filtrations: the first, $F^\bullet$ ("de Rham filtration"), arises from that of $H^i_{dR}$ via the extension of scalars isomorphism:

$$H^i_{dR}(X) \otimes_K \mathbb{C} \xrightarrow{\sim} H^i(X_{\sigma}(\mathbb{C}), \mathbb{C}) \xrightarrow{\sim} H^i(X_{\sigma}(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$ 

The second filtration $\overline{F}^\bullet$ comes from the first via complex conjugation. Together, they provide the Hodge filtration:

$$F^p H^i(X_{\sigma}(\mathbb{C}), \mathbb{C}) \cap \overline{F}^q H^i(X_{\sigma}(\mathbb{C}), \mathbb{C})$$
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Finally, we have

3. \( \ell \)-adic Étale Cohomology. With \( \ell \) a rational prime as above, we have a sequence of \( \mathbb{Q}_\ell \)-vector spaces \( H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \), the \( \ell \)-adic cohomology of \( X_{\overline{K}} := X \otimes_K \overline{K} \).

Important: \( G = \text{Gal}(\overline{K}/K) \) acts on \( X_{\overline{K}} \), and hence on \( H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \); in particular, so does \( \text{Frob}_p \) for any prime \( p \notin S \).
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Interrelationships

Given $\overline{\sigma} : \overline{K} \rightarrow \mathbb{C}$ which extends $\sigma$, and an embedding $\mathbb{Q}_\ell \rightarrow \mathbb{C}$, we have canonical isomorphisms

\[
(*) \quad H^i(X_{\overline{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \sim H^i(X_{\sigma}(\mathbb{C}), \mathbb{C}) \sim H^i_{dR}(X) \otimes_K \mathbb{C}.
\]

These permit the transfer of information from each setting to the others.

Weights

Each of the 3 cohomology theories (independently) carries an increasing weight filtration $W_\bullet$ (due to Deligne).

The isomorphisms $(*)$ above respect the weight filtrations.
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(*) \ H_i^j(X_{\bar{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{C} \sim H_i^j(X_{\sigma}(\mathbb{C}), \mathbb{C}) \sim H_{dR}^i(X) \otimes_K \mathbb{C}.
\]

These permit the transfer of information from each setting to the others.

Weights

Each of the 3 cohomology theories (independently) carries an increasing weight filtration $W_\bullet$ (due to Deligne).

The isomorphisms (*) above respect the weight filtrations.
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The 3 filtrations $F^\bullet$, $\bar{F}^\bullet$, and $W_\bullet$ all interact in $H^i(X_\sigma(\mathbb{C}), \mathbb{C})$ in a way which connects the 3 cohomology theories.

We have:

$$F^p H^i(X_\sigma(\mathbb{C}), \mathbb{C}) \cap \bar{F}^q H^i(X_\sigma(\mathbb{C}), \mathbb{C}) \subset W_{p+q} H^i(X_\sigma(\mathbb{C}), \mathbb{C})$$

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Theorem.

(Kisin-L P&AMQ (Coates issue) 2006) Let $K, S$ etc. be as above, and let $X$ be a variety over $K$. Assume that for each prime $p \notin S$, the eigenvalues of $\text{Frob}_p$ on $H^j(X, \mathbb{Q}_\ell)$ are all of the form $\zeta q_p^i$ ($i \in \mathbb{N}, \zeta$ a root of unity.), and that for any $i \in \mathbb{N}$, there are $r_i$ of these.

Then $\text{Gr}_F^p \text{Gr}_F^q H^j(X_\sigma(\mathbb{C}), \mathbb{C})$ has dimension $r_i$ if $p = q = i$, and is 0 otherwise.

NB The hypothesis is about eigenvalues of Frobenius, while the conclusion is about the Hodge filtration, which does not exist in $\ell$-adic cohomology.

Say that $X$ is mixed Tate (mt) if it satisfies the conditions of the theorem.

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A good source of mt varieties is:

**Prop:** $p : X \rightarrow Y$ a smooth morphism of smooth $K$-varieties such that each fibre $p^{-1}(y)$ is $K$-isomorphic to a fixed $Z$. Assume that the local systems $R^j p_* \mathbb{C}$ induced by $p : X_\sigma(\mathbb{C}) \rightarrow Y_\sigma(\mathbb{C})$ are constant for each $j$. If any 2 of $X$, $Y$, $Z$ are mt then so is the third.
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The following consequence is relevant to toric varieties.

Prop Suppose $X$ is such that for almost all $q$, $\text{Frob}_q$ has eigenvalues of absolute value $q^{i/2}$ on $H^i_c(X_{\bar{K}}, \bar{\mathbb{Q}}_{\ell})$. Then the following are equivalent:

(1) $X$ is mt.
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Toric Varieties

Given a lattice $N \subset \mathbb{R}^n$, a rational convex polyhedral cone (cpc) is a set of the form $\sigma := \{ \sum_{i=1}^{\ell} \lambda_i a_i \mid \lambda_i \geq 0 \}$, where the $a_i \in N$

Corresponding to $\sigma$ there is an affine toric variety $T(\sigma)$, defined in terms of the dual lattice $M := N^\vee = \{ u \in (\mathbb{R}^n)^* \mid (v, u) \in \mathbb{Z} \ \forall \ v \in N \}$

If $\tau$ is a face of $\sigma$, then $T(\tau) \hookrightarrow T(\sigma)$.

Generally, if $\Delta$ is a fan of cpc’s, $T(\Delta) = \bigcup_{\sigma \in \Delta} T(\sigma)$,

with suitable identifications arising from the face relations.
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Generally, if \( \Delta \) is a fan of cpc’s, \( T(\Delta) = \bigcup_{\sigma \in \Delta} T(\sigma) \), with suitable identifications arising from the face relations.
The toric variety associated with a root system.

Let $\Phi \subset \mathbb{R}^n := V$ be a crystallographic root system; take $N$ to be the weight lattice, and $\Delta$ the set of cones (‘regions’) into which $V$ is divided by the hyperplanes orthogonal to the roots.

Assume chosen a simple system $\Pi \subset \Phi$, and write $W$ for the Weyl group of $\Phi$.

The cones are then in bijection with the set of cosets $wW_J$ ($w \in W, J \subseteq \Pi$).

The corresponding toric variety is defined over $\mathbb{Z}$ and is denoted $\mathcal{T}_W$.

It has a $W$-action, and we’ll see how the method of rational points gives a very simple formula for $P_{\mathcal{T}_W}(w, t)$ ($w \in W$).
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It has a \( W \)-action, and we’ll see how the method of rational points gives a very simple formula for \( P_{T_w}(w, t) \) (\( w \in W \)).
Properties of the $T(\Delta)$.

- $T(\Delta)$ is complete iff $\bigcup_{\sigma \in \Delta} \sigma = V$. Thus $T_W$ is complete.

- $T_W$ is also non-singular, because each $\sigma \cap \mathbb{Z}\Phi$ is generated by part of a basis of $\mathbb{Z}\Phi$.

- The torus $T = \mathbb{C}^\times = T(\{0\})$ acts on $T_W$, with a dense orbit (itself).

- In general, each affine piece $T(\sigma)$ has a distinguished point $P_{\sigma}$ whose $T$-orbit $O_{\sigma}$ is a torus of dimension $n - \dim \sigma$. If $\sigma \sim wW_J$, write $O(wW_J)$ for the corresponding orbit; $O(wW_J) \simeq (\mathbb{C}^\times)^{|J|}$.

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- The torus $T = \mathbb{C}^\times = \mathcal{T}({\{0\}})$ acts on $T_W$, with a dense orbit (itself).
- In general, each affine piece $\mathcal{T}(\sigma)$ has a distinguished point $P_\sigma$ whose $T$-orbit $O_\sigma$ is a torus of dimension $n - \dim \sigma$. If $\sigma \sim wW_J$, write $O(wW_J)$ for the corresponding orbit; $O(wW_J) \simeq (\mathbb{C}^\times)^{|J|}$.
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$W$ permutes the orbits $O(wW_j)$: $xO(wW_j) = O(xwW_j)$.

**Proposition.** $\mathcal{I}_W$ is mixed Tate, has only even non-zero cohomology, and $\text{Frob}_q$ acts on $H^{2i}(\mathcal{I}_W, \mathbb{Q}_\ell)$ with all eigenvalues equal to $q^i$.

Proof: Fulton observed that in general, $|\mathcal{I}(\Delta)(\mathbb{F}_q)| = \sum_{\sigma \in \Delta} (q - 1)^{n - \dim \sigma}$, which is a polynomial in $q$. Since in our case $\mathcal{I}_W$ is non-singular and projective, the eigenvalues of $\text{Frob}$ on $H^i$ all have absolute value $q^{i/2}$. The proposition now follows by our earlier result.

Also require the standard result: if $x \in W$, then $|\mathcal{T}(\bar{\mathbb{F}}_q)^{\text{Frob}_q}| = \det_V(q - x)$.

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**Theorem.** For $J \subseteq \Pi$, let $\gamma_J : W_J \to \mathbb{Z}[t]$ be the function $\gamma_J(w) = \det_{V_J}(t^2 - w)$. Then $P_{\mathcal{T}_W}(w, t) = \sum_i \text{trace}(w, H^i(\mathcal{T}_W, \mathbb{C}))t^i$ is given by

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- $(H^*(\mathcal{T}_W), 1)_W = (1 + t^2)^n$ (well known)
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The case $t = 1$

In 1968 (or earlier) R. Steinberg proved

Let $\tilde{\Pi} = \Pi \Pi \{ -\alpha_h \}$, where $\alpha_h$ is the highest root. Then writing $r = |\Pi|,$

$$\det v(1 - w) = \sum_{J \subseteq \tilde{\Pi}} (-1)^r |J| \text{Ind}_{W_J}^W(1)$$

This may be used to express $H^* (\mathcal{T}_W)$ as a sum of permutation characters (cf. Stembridge (1995)).

In type $A$ one can do better: one has the following “$q$-analogue” of Steinberg’s formula:

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This may be used to express the result for type A in terms of the ring $\Lambda[[t]]$, where $\Lambda = \bigoplus_{n \geq 0} R_n$, and $R_n$ is the character ring of $\text{Sym}_n$.

Write $\tau_{n,q} = \sum_i \text{trace}(-, H^{2i}(T_W)) q^i \in R_n[q]$. Then

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Varieties over \(\mathbb{R}\)–the real case.

When \(K \subset \mathbb{R}\), the variety \(X\) also has an associated manifold \(X(\mathbb{R})\) of real points.

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Varieties over $\mathbb{R}$–the real case.

When $K \subset \mathbb{R}$, the variety $X$ also has an associated manifold $X(\mathbb{R})$ of real points.

If $X$ is mt, then $P_X(-1)$ is related to the Euler characteristic of $X(\mathbb{R})$ in general.
The real case $\mathcal{T}(\mathbb{R})$

Since the $\mathcal{T}_W$ are defined over $\mathbb{Z}$, we may consider their real points $\mathcal{T}(\mathbb{R})$. These are compact connected smooth real manifolds.

In joint work with A. Henderson, we do not yet have the graded representation of $W$ on $H^*(\mathcal{T}_W(\mathbb{R}))$, but we do have the equivariant Euler characteristic, which shows that the situation is quite different from the complex points.

Define the generalised character $\psi_W \in R(W)$ by

$$\psi_W(w) = (-1)^r \varepsilon(w) \pi^{(2)}_W(w),$$

where $\varepsilon$ is the sign character of $W$, and $\pi^{(2)}_W$ is the permutation character of $W$ on the finite set $N/2N$ ($N=$wt lattice).

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$$\Lambda_W = \sum_{J \subseteq \Pi} \text{Ind}_{W_J}^W \psi_J,$$

**EXAMPLE:** $W = \text{Sym}_n$

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