Lusztig's Conjecture

$k = \bar{k}$ field

G simply conn. alg. group / k

$T \subseteq B \subseteq G$ max torus, Borel subgroup

$X = \mathrm{Hom}(T, k^*) \supseteq X^+$ dominant weights

$\lambda \in X^+ \rightsquigarrow L(\lambda)$ simple rational rep of G with highest weight λ.

$L(\lambda)_\mu$ = μ weight space with rep T, $\mu \in X$

$[L(\lambda)] = \sum_{\mu \in X} \dim_k L(\lambda)_\mu e^\mu \in \mathbb{Z}[X]$

Problem $[L(\lambda)] = ?$

When char $k = 0$, Weyl's character formula

$$[L(\lambda)] = \chi(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}}$$

W Weyl gp, $l : W \to \mathbb{N}$ length, $\rho = \frac{1}{2} \sum_{\substack{\text{pos roots} \\ \alpha}} \alpha$

What about char $k = p > 0$?

let $\hat{W}$ be affine Weyl gp

$w \in \hat{W}$ is dominant if $0 < \langle w(\rho), \alpha^\vee \rangle$ $\forall \alpha$ simple

restricted if $0 < \langle w(\rho), \alpha^\vee \rangle < p$

There exists a regular $\hat{W}$-orbit in X iff $p > h$

Coxeter number.

We assume this from now on.

Lusztig's conjecture '79

If w is restricted, then

$$[L(w(p)-\rho)] = \sum_{\substack{x \in \hat{W} \\ \text{dominant}}} (-1)^{l(w)-l(x)} h_{w_0 x, w_0 w}(1)\ \chi(x(p)-\rho)$$

$w_0 \in W$ longest element

$h_{\cdot,\cdot}$ affine Kazhdan-Lusztig polynomial.

$\mathfrak{g} = \mathrm{Lie}\ G \rightsquigarrow Z'(\lambda)$ baby Verma module $\lambda \in X$

$L'(\lambda)$ simple quotient

Conjecture $[Z'(w\cdot 0) : L'(x\cdot 0)] = p_{w_0 w, w_0 x}(1)$

$w\cdot\mu = w(\mu+\rho)-\rho$; periodic polynomial

$\Updownarrow$

Lusztig's ~~Thm~~ Conjecture.

Theorem ('93-95) (KT & KL & AJS)

Conjecture is true for $p \gg 0$

Theorem ('07) (F & AJS)

Conjecture is true for $p \gg 0$

$[Z'(w\cdot 0) : L'(x\cdot 0)] = 1$ iff $P_{w_0 w, w_0 x}(1) = 1$ $\forall p \geq h$.

Sketch of proof

$$\begin{tikzcd}
\text{intersection cohomology cplx with coeffs on } \mathcal{Fl} \arrow[r, "KT"] & \text{repn of affine KM-alg} \arrow[r, "KL"] & \text{reps of big quantum gp} \arrow[r] & \text{reps of small quantum gp} \\
\text{sheaves on } k\text{-vector sp on } \mathcal{Fl}^\vee \arrow[r] & \text{sheaves on moment graphs} \arrow[r] & \text{AJS combinatorics} \arrow[ur] \arrow[r] & \text{modular rep} \arrow[d] \\
 & & & \text{Lusztig's Conjecture}
\end{tikzcd}$$

Sheaves on affine flag varieties

$\mathcal{Fl}^\vee$ - complex affine flag variety for dual root system

$\hat{T}^\vee = T^\vee \times \mathbb{C}^\times$ extended torus acts on $\mathcal{Fl}^\vee$

Simple affine refl in $\hat{W}$ $\rightsquigarrow$ $D_{\hat{T}}(\mathcal{Fl}^\vee, k)$ equivariant derived category of sheaves of k-vec spaces on $\mathcal{Fl}^\vee$

$s \in \hat{\mathcal{S}} \rightsquigarrow \mathcal{Fl}^\vee_s$ partial affine flag variety

$$\pi_s : \mathcal{Fl}^\vee \to \mathcal{Fl}^\vee_s$$

Def: $\mathcal{J}_k = \left\{ \begin{array}{l} \text{direct sums of direct summands of object} \quad s,\dots,t \in \hat{\mathcal{S}} \\ \pi_s^* \circ \pi_{s*} \circ \dots \circ \pi_t^* \circ \pi_{t*} \underline{k}_p \end{array} \right\} \subseteq D_{\hat{T}}(\mathcal{Fl}^\vee, k)$

$\underline{k}_p$ skyscraper on basepoint

Equivariant hypercohomology

$(\mathcal{Fl}^\vee)^{\hat{T}^\vee}$ = fixed points $\cong \hat{W}$

$i_w : \{pt\} \hookrightarrow \mathcal{Fl}^\vee$ inclusion of fixed pt for $w \in \hat{W}$

$$F \in \mathcal{J}_k \rightsquigarrow \mathbb{H}^*_{\hat{T}}(F) \longrightarrow \bigoplus_{w \in \hat{W}} \underbrace{\mathbb{H}^*_{\hat{T}}(i_w^* F)}$$

$\hat{S} = S_k((X^\vee \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} k)$ - modules

On the representation theoretic side

$S = S((\mathrm{Lie}\, T)^*) \rightsquigarrow Z'_S(\lambda)$ family of baby Verma modules

$P_S(\lambda)$ - family of projectives

R_k = {deformed rep of $\mathfrak{g}$ that admit a Verma flag}

$P'_S(\lambda) \in R_k$

$$(P'_S(\lambda) : Z'_S(\mu)) = (P'(\lambda) : Z'(\mu)) = [Z'(\mu) : L'(\lambda)] .$$

Theorem: There is a functor $\Phi : \mathcal{J}_k \longrightarrow R_k$

such that $\left(\Phi(F) : Z'_s(x\cdot o)\right) = \operatorname{rk} \mathbb{H}^*_{\hat{T}}(i_x^* F)$

Proof of Lusztig's conjecture

- show that for w restricted there is $F_w \in \mathcal{J}_k$ with $\Phi(F_w) \cong P'_s(w\cdot o)$
- show for almost all $p >$ F_w is equivariant intersection cohomology complex
- Kazhdan-Lusztig $\operatorname{rk} \mathbb{H}^*_{\hat{T}}(i_x^* F_w) = h_{x,w}(1)$
  $p \gg 0$ $\qquad = P_{w_0 x, w_0 w}(1)$

Remark: $\mathcal{J}_k$ - categorification of affine Hecke alg. $\mathcal{H}$

R_k - categorification of the periodic module $\mathcal{M}$

$\varphi : \mathcal{H} \longrightarrow \mathcal{M}$

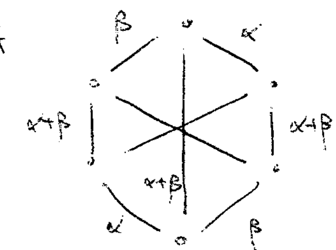
$a \mapsto a \cdot A_o$

fundamental alcove

Φ is a categorification of φ

$\mathbb{H}^*_{\hat{T}}(F) \longrightarrow \bigoplus_{w \in \hat{W}} \mathbb{H}^*_{\hat{T}}(i_w^* F)$

A_2 moment graph



Sheaf on moment graphs

$\mathcal{M} = (\mathcal{M}^x, \mathcal{M}^E, S_{x,E})$

$\hat{S}$-module x vertex

$\hat{S}$-module for $E : x \xrightarrow{\alpha} y$ with $\alpha \mathcal{M}^E = 0$

$S_{x,E} : \mathcal{M}^x \longrightarrow \mathcal{M}^E$

$\rightsquigarrow$ global sections $\Gamma(\mathcal{M}) = \left\{ (m_x) \in \prod \mathcal{M}^x \;\middle|\; \begin{array}{c} S_{x,E}(m_x) = S_{y,E}(m_y) \\ E : x \longrightarrow y \end{array} \right\}$.

$\square$