k alg. closed field of char $p \gg 0$

G simple s/c alg. k-gp

$\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{g}$ is restricted, $x \mapsto x^{[p]}$

$U = U(\mathfrak{g})$ the enveloping alg. of $\mathfrak{g}$

$Z_p(\mathfrak{g}) = \langle x^p - x^{[p]},\ x \in \mathfrak{g} \rangle$, a polynom alg in $\dim \mathfrak{g}$ variables

Pick $\chi \in \mathfrak{g}^*$ and set $U_\chi(\mathfrak{g}) = U(\mathfrak{g}) / I_\chi$

$I_\chi = \langle x^p - x^{[p]} - \chi(x)^p,\ x \in \mathfrak{g} \rangle$

$(,)$ Killing form

Assume $\chi = (e, \cdot)$, e nonzero nilp element in $\mathfrak{g}$.

$\exists$ cocharacter $\lambda_e : k^* \to G$ s.t.

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \qquad e \in \mathfrak{g}(2)$$

$$\mathfrak{g}(e) = \{ x \in \mathfrak{g} \mid [x, e] = 0 \} \subseteq \bigoplus_{i \geq 0} \mathfrak{g}(i)$$

We have a nondeg. skew-sym form on $\mathfrak{g}(-1)$.

$$\langle x, y \rangle = (e, [x, y]) \qquad \forall\ x, y \in \mathfrak{g}(-1).$$

$\mathfrak{g}(-1)^\circ$ max. totally isotrop. subspace for $\langle , \rangle$ and set

$\mathfrak{g}(-1)^\circ \subset \mathfrak{g}(-1)$

$$\mathfrak{m} := \mathfrak{g}(-1)^\circ \bigoplus_{i \geq 2} \mathfrak{g}(-i)$$

Then $\dim \mathfrak{m} = \frac{1}{2} \dim G \cdot e$ and χ vanishes on $[\mathfrak{m}, \mathfrak{m}]$.

So $\chi : \mathfrak{m} \to k$ is a rep. (k_χ the corr. module)

Define $Q_\chi^{[p]} = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{m})} k_\chi$,

$$W = W^{[p]}(\mathfrak{g}, e) = (\mathrm{End}_{\mathfrak{g}}\, Q_\chi^{[p]})^{op}$$

It turns out that $\underbrace{Q_\chi^{[p]} \oplus \cdots \oplus Q_\chi^{[p]}}_{p^{d(e)}} \cong U_\chi(g)$

$d(e) = \frac{1}{2} \dim G \cdot e$

Then $U_\chi(g) \cong \mathrm{Mat}_{p^{d(e)}}(W)$.

Hence every f.d. $U_\chi(g)$ - module $p^{d(e)}$ has dim divisible by $p^{d(e)}$.

Problem of smallest rep: Does there always exist a $U_\chi(g)$-module of dim $p^{d(e)}$?

1. Suppose e is Richardson

$$g = n_- \oplus \underbrace{l \oplus n_+}_{p} \qquad p = \mathrm{Lie}\ P$$

$e \in n_+$, $\bar{P} \cdot e = n_+$, Note $U_\chi(p) \cong U^{[p]}[p]$

Take a 1-dim. l-module, say k_λ, and consider $U_\chi(g) \otimes_{U_\chi(p)} k_\lambda$.

This has dim $= p^{\dim u} = p^{d(e)}$.

2. Suppose e is induced from $e_0 \in l$

$e = e_0 + e_1$ where e_1 is Richardson in n_+.

Suppose $U_{\chi_0}(l)$ has a module of dim $p^{d(e_0)}$, say V.

$\chi_0 = (e_0, \cdot)$.

Then consider $U_\chi(g) \otimes_{U_\chi(p)} V := \tilde{V}$

Then $\dim \tilde{V} = p^{d(e)}$.

Give $U(\mathfrak{g})$ Kazhdan filtration by setting

$$\deg_i(x) = i+2 \quad \text{if} \quad x \in \mathfrak{g}(i).$$

Let $x_1, \dots, x_r$ be a basis of $\mathfrak{g}_e$ s.t.

$$x_i \in \mathfrak{g}(n_i), \quad n_i \in \mathbb{Z}_+ .$$

Then W has k-basis $\{ \theta_1^{a_1} \cdots \theta_r^{a_r} \mid 0 \le a_i \le p-1 \}$

$$\theta_i(1_\chi) = \Big(x_i + \underset{\text{of } \chi \text{ degree } n_i+2}{\text{nonlin. terms}} + \text{lower terms} \Big) \otimes 1_\chi$$

Consider $Q_\chi = U(\mathfrak{g}_\mathbb{C}) \otimes_{U(m_\mathbb{C})} \mathbb{C}_\chi$,

$$W_\mathbb{C} := W(\mathfrak{g}_\mathbb{C}, e) = (\operatorname{End}_{\mathfrak{g}_\mathbb{C}} Q_\chi)^{op}$$

↑ finite W-alg. assoc with $(\mathfrak{g}_\mathbb{C}, e)$.

It has PBW basis $\{ \theta_1^{a_1} \cdots \theta_r^{a_r} \mid a_i \in \mathbb{Z}_+ \}$.

$Z(\mathfrak{g}_\mathbb{C})$ center of $U(\mathfrak{g}_\mathbb{C})$

Some properties.

① $Z(\mathfrak{g}_\mathbb{C}) \hookrightarrow W_\mathbb{C}$ and $Z(\mathfrak{g}_\mathbb{C}) = Z(W_\mathbb{C})$

② $W_\mathbb{C}$ is a free $Z(\mathfrak{g}_\mathbb{C})$-module

③ There are polynomials F_{ij}, $1 \le i < j \le r$, over $\mathbb{Q}$ s.t.

(*) $[\theta_i, \theta_j] = F_{ij}(\theta_1, \dots, \theta_r)$ is a presentation of $W_\mathbb{C}$.

④ Every $W_\mathbb{C}$ has f.d. irr. representation.
Moreover, $\forall\, u \in W_\mathbb{C}$, $\exists$ a f.d. rep ρ of $W_\mathbb{C}$ s.t. $\rho(u) \neq 0$

⑤ $W_\mathbb{C}$ is a deformation of $U(\mathfrak{g}_{\mathbb{C},e})$

⑥ For every central char $\eta: Z(\mathfrak{g}_\mathbb{C}) \longrightarrow \mathbb{C}$ the algebra $W_\mathbb{C} \otimes_{Z(\mathfrak{g}_\mathbb{C})} \mathbb{C}_\eta$ is a deformation of $\mathbb{C}[S_e \cap \mathcal{N}(\mathfrak{g}_\mathbb{C})]$; $S_e = e + \mathfrak{g}_{\mathbb{C},f}$, (e,h,f) $\mathfrak{sl}_2$-triple.

$W_{\mathbb{C}}$ is defined over $\mathbb{Z}[N^{-1}] =: A$,

$N = N(e) \in \mathbb{Z}_+$

Can define W_A, a free A-module.

Can define $W_k = W_A \otimes_A k$ $\quad p \gg 0$

Then W_k has k-basis $\{ \theta_1^{a_1} \cdots \theta_r^{a_r} \mid a_i \in \mathbb{Z}_+ \}$

There is an algebra epimorphism $\phi : W_k \longrightarrow W^{[p]}(g, e)$

Ex. If e is reg. nilpotent, then

$W_k = F[x_1, \ldots, x_\ell]$, $\quad \ell = rk(g)$;

but $W^{[p]}(g, e)$ is a direct sum of partial coinvariant algebras.

Thm 1. if $W(g_{\mathbb{C}}, e) = W_{\mathbb{C}}$ has 1-dim representations, then $W^{[p]}(g, e)$ has 1-dim representations.

By using a result of McGovern, Losev proved that $W(g_{\mathbb{C}}, e)$ has 1-dim representations if $g_{\mathbb{C}}$ is of type B, C, D.

Corollary: if $p \gg 0$ and g of type B, C, D, then $U_\chi(g)$ has a module of dim $p^{d(e)}$, $\chi = (e, \cdot)$.

Conj: Every finite W-algebra $W(g_{\mathbb{C}}, e)$ has a 1-dim rep.

Thm 2: Suppose e is induced from $e_0 \in \ell$.

If $W([\ell_{\mathbb{C}}, \ell_{\mathbb{C}}], e_0)$ has 1-dim representations, then $W(g_{\mathbb{C}}, e)$ has 1-dim representations.

Sketch: $W([l_{\mathbb{C}}, l_{\mathbb{C}}], e)$ has 1-dim reps iff $V(F_{ij}([l_{\mathbb{C}}, l_{\mathbb{C}}]))$ has a pt in a ring of S-integers in an alg. number field.
Then $W([l, l], e_c)$ has a 1-dim rep.
From this, $U_{\chi_0}(l)$ has a module of dim $p^{d(e_c)}$.
By our earlier example, $U_\chi(g)$ has a module of dim $p^{d(e)}$. Then $W^{[p]}(g, e)$ has 1-dim rep.
Then $V(F_{ij}(g)) \neq \emptyset$ for $p \gg 0$.
Then $V(F_{ij}(g_{\mathbb{C}})) \neq \emptyset$. ∎

For $g_{\mathbb{C}}$, $\mathcal{C}_\chi$ = all $g_{\mathbb{C}}$-modules V s.t. $x - \chi(x)$ acts on V locally nilp. $\forall\, x \in m_{\mathbb{C}}$.

$Wh_\chi(V) = \{ v \in V \mid x \cdot v = \chi(x) v \quad \forall\, x \in m \}$

$W_{\mathbb{C}}\text{-mod} \ni M \longmapsto Q_\chi \otimes_{W_{\mathbb{C}}} M \in \mathcal{C}_\chi$.

Skryabin: There are equivalences.

If M is a f.d. irr. $W_{\mathbb{C}}$-module, then $Q_\chi \otimes_{W_{\mathbb{C}}} M$ is irr, hence $I_M := \mathrm{Ann}_{U_{\mathbb{C}}}(Q_\chi \otimes_{W_{\mathbb{C}}} M)$ is a prim. ideal.

One knows: $VA(I_M) = \overline{G_{\mathbb{C}} e}$.

Conversely (P, Losev) every prim. ideal I with $VA(I) = \overline{G_{\mathbb{C}} e}$ occurs this way.

We have a map: $\mathbb{O} = G_{\mathbb{C}} e$

$$\mathcal{X}^\lambda_{\mathbb{O}} \twoheadleftarrow \mathrm{Irr}_\lambda W_{\mathbb{C}}$$

$\mathcal{X}^\lambda_{\mathbb{O}}$: all prim ideals with $VA = \overline{\mathbb{O}}$ and infinitesimal char λ.

$\mathrm{Irr}_\lambda W_{\mathbb{C}}$: (all irr. f.d. $W_{\mathbb{C}}$-modules with central char λ) / iso

This map is finite-to-one.

Conjecture: The fibres of this map are the orbits under the action of $\Gamma(e) = (G_\mathbb{C})_e / (G_\mathbb{C})_e^\circ$.