1. Deform quantization

Fix field $\mathbb{C}$

X a smooth affine alg. variety

ω is a symplectic form (nondegenerate, $d\omega = 0$)

G a reductive gp $G : X$, G preserves ω

$\mathbb{C}^\times = \{ z \in \mathbb{C} \mid z \neq 0 \}$ $\mathbb{C}^\times : X$

1. $\mathbb{C}^\times$ commutes with G
2. $t \cdot \omega = t^k \omega$ $k \in \mathbb{Z}_{>0}$

Ex. 1. X a symplectic vector space

ω is constant

$G = Sp(X)$

$\mathbb{C}^\times : X$, $t \cdot x = t^{-1} x$, $k = 2$

Ex 2. X_0 smooth affine variety, $G : X_0$, $X = T^* X_0$,

ω is canonical symplectic form

$G X_0 \rightsquigarrow G : X$

$\mathbb{C}^\times : X$ fiberwise dilations $t \cdot (x_0, \beta) = (x_0 ; t^{-1} \beta)$

$t \in \mathbb{C}^\times$ $k = 1$.

Def: A star-product $*$ on X is $* : \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[X] \to \mathbb{C}[X][[\hbar]]$

s.t. 1. $1 * f = f * 1 = f$ $\forall f \in \mathbb{C}[X]$

2. $(f * g) * h = f * (g * h)$

$$\left(\sum_{i=0}^{\infty} f_i \hbar^i \right) * \left(\sum_{j=0}^{\infty} g_j \hbar^j \right) = \sum_{i,j=0}^{\infty} (f_i * g_j) \hbar^{i+j}$$

3. $f * g \equiv fg \pmod{\hbar}$ $\forall f, g \in \mathbb{C}[X]$

4. $\{\cdot, \cdot\} : \mathbb{C}[X] \wedge \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

$$f * g - g * f \equiv \hbar \{f, g\} \pmod{\hbar^2}.$$

$f * g = \sum_{i=0}^{\infty} D_i(f, g) \hbar^i \qquad D_i : \mathbb{C}[X] \otimes \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

3. $\Leftrightarrow \quad D_0(f, g) = fg$

4. $\Leftrightarrow \quad D_1(f, g) - D_1(g, f) = \{f, g\}$

The action of $G \times \mathbb{C}^\times$: $\mathbb{C}[X][[\hbar]]$:

$$g \sum_{i=0}^{\infty} f_i \hbar^i = \sum_{i=0}^{\infty} (g f_i) \hbar^i \qquad \forall g \in G$$

$$t \sum_{i=0}^{\infty} f_i \hbar^i = \sum_{i=0}^{\infty} (t f_i) t^{k i} \hbar^i \qquad \forall t \in \mathbb{C}^\times$$

$*$ should be $G \times \mathbb{C}^\times$ - equivariant.

Fedosov construction

Def. A connection ∇ on X is called symplectic if it is an affine conn. torsion-free $\nabla \omega = 0$.

Claim: There is a $G \times \mathbb{C}^\times$ - invariant symplectic conn. ∇ on X.

$\nabla \rightsquigarrow *$, if ∇ is $G \times \mathbb{C}^\times$ - invariant, then $*$ is $G \times \mathbb{C}^\times$ - equiv.

1. $*$ is differential, D_i are bidifferential operators

2. $*$ does not depend on ∇ up to an isomorphism.

$\nabla \rightsquigarrow *$

$\nabla' \rightsquigarrow *'$ $\quad \Rightarrow \exists \; T : \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$ diff. op s.t. $T_0 = id$

$T = \sum_{i=0}^{\infty} T_i \hbar^i$ is an iso

$(Tf) *' (Tg) = T(f * g)$

T_i are $G \times \mathbb{C}^\times$ - equiv.

Ex. 1 ∇ is the canonical flat connection

$*$ is the Moyal-Weyl on $\mathbb{C}[X]$

$\mathbb{C}[X] * \mathbb{C}[X] \subseteq \mathbb{C}[X][\hbar]$

$f *_1 g = \sum_{i=0}^{\infty} D_i(f, g)$, $\quad \mathbb{C}[X] \cong \underbrace{A_X}_{\text{Weyl algebra}} = T(X) / (x \otimes y - y \otimes x - \omega(x, y))$

Ex 2. $\nabla = G \times \mathbb{C}^\times$ invar. connection $\rightsquigarrow$ $*$

$\mathbb{C}[X] = \bigoplus_{i \geq 0} \mathbb{C}[X]_i$, D_j decreases degrees by j

$f * g = \sum_{i=0}^{\infty} D_i(f,g)\hbar^i \in \mathbb{C}[X][\hbar] \rightsquigarrow f *_1 g = \sum_{i \geq 0}^{\infty} D_i(f,g)$

$\mathbb{C}[X], *_1 \cong D(X_0)$.

New example: Equivariant Slodowy slice.

G is semisimple, $\mathfrak{g} = \mathrm{Lie}(G)$, $e \in \mathfrak{g}$ a nilp. element.

(e,h,f) an $\mathfrak{sl}_2$-triple, $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$.

$\gamma : \mathbb{C}^\times \longrightarrow G$, $\frac{d}{dt}\big|_{t=0} \gamma = h$,

$(\cdot,\cdot)$ Killing form $\mathfrak{g} \cong \mathfrak{g}^*$

$S = e + \mathfrak{z}_{\mathfrak{g}}(f)$ $\{\xi, [\xi,f]=0\}$

Def. (Equiv. sl slice) $X = G \times S$

$G \curvearrowright X$ $g \cdot (g_1, s) = (gg_1, s)$

$X \subset G \times \mathfrak{g} \cong G \times \mathfrak{g}^* = T^*G$

Restrict the symplectic form to X, 2-form ω on X.

$\mathbb{C}^\times : T^*G$ $t \cdot (g,s) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)s)$ $g \in G$ $s \in \mathfrak{g}$

Properties of $\mathbb{C}^\times : T^*G$

1. $\mathbb{C}^\times$ commutes with G, $\mathbb{C}^\times$ preserves X, $G(1,e)$
2. $t \cdot \omega = t^2 \cdot \omega$
3. $S = X/G \rightsquigarrow \mathbb{C}^\times : S$. the grading on $\mathbb{C}[S]$ is positive

$\Updownarrow$

$\lim_{t \to \infty} ts = e$

$\omega_{(1,e)}$ is nondegenerate

$\Rightarrow$ [1-5] ω is nondeg. on X.

∇ a $G \times \mathbb{C}^\times$ invariant connection on X

$\downarrow$

$* : \mathbb{C}[X] \otimes \mathbb{C}[X] \longrightarrow \mathbb{C}[X][[\hbar]]$

$\mathbb{C}[X] \otimes \mathbb{C}[X] \subseteq \mathbb{C}[X][\hbar]$ ($\Leftarrow$ grading $\mathbb{C}[S]$ is positive)

$\mathbb{C}[X], *_1$ - the equivariant W-algebra $\widetilde{\mathcal{W}}$

$G : \widetilde{\mathcal{W}}$, $F_i \widetilde{\mathcal{W}} = \bigoplus_{j \le i} \mathbb{C}[X]_j$ filtration

$\operatorname{gr} \widetilde{\mathcal{W}} = \mathbb{C}[X]$

Def: W-alg $\mathcal{W} = \widetilde{\mathcal{W}}^G$, $\operatorname{gr} \mathcal{W} = \mathbb{C}[S]$

Thm (IL) Suppose that $\mathcal{W}$ has a fin. dim. module (proved by Premet, IL). Then there is a faithful locally fin. dim $\mathcal{W}$-module.

Proof 1. $\widetilde{\mathcal{W}}$ is simple ($\Leftarrow \operatorname{gr} \widetilde{\mathcal{W}} = \mathbb{C}[X]$)

$\Downarrow$

any $\widetilde{\mathcal{W}}$-module is faithful

2. $\widetilde{\mathcal{W}} \cong_G \mathbb{C}[G] \otimes \mathcal{W}$ as right $\mathcal{W}$-modules

$\mathbb{C}[X] = \mathbb{C}[G] \otimes \mathbb{C}[S]$.

V a fin. dim $\mathcal{W}$-module

$\widetilde{V} = \widetilde{\mathcal{W}} \otimes_{\mathcal{W}} V \cong_G \mathbb{C}[G] \otimes V$

$= \bigoplus_{\lambda \in \operatorname{Irrep} G} \mathbb{C}[G]_\lambda \otimes V$

$\mathcal{W}$-module, faithful

$\mathcal{W}$-submodule.

$\dim \mathbb{C}[G]_\lambda \otimes V < \infty$. $\square$

Partial Classification of fin dim irred. $\mathcal{W}$-irred.

To compare two sets

1. $I_W = \{$ irred. fin. dim. $\mathcal{W}$-modules $\}/\sim$

   $= \{$ primitive ideals of fin. codim $\}$

2. $I_U = \{$ primitive ideals $\mathcal{I} \subset U(\mathfrak{g}) \mid V(\mathcal{I}) = \overline{Ge} \}$

   There is a surjective map $I_W \twoheadrightarrow I_U$ with finite fibers.

1. $\tilde{I}_W = \{$ all two-sided ideals of fin. codim $\}$
2. $\tilde{I}_U = \{ \mathcal{I} \subset U(\mathfrak{g}) \mid$ two-sided ideals with $V(\mathcal{I}) = \overline{Ge} \}$

   $\mathcal{I} \longmapsto \mathcal{I}^+ : \tilde{I}_W \longrightarrow \tilde{I}_U$

   $\mathcal{J} \longmapsto \mathcal{J}_+ : \tilde{I}_U \longrightarrow \tilde{I}_W$.

Thm (IL)

1. $\mathcal{I}^+$ is prime $\Leftarrow$ $\mathcal{I}$ is.

   Goldie rk $U(\mathfrak{g})/\mathcal{I}^+ \leq$ Goldie rk $W/\mathcal{I} = \dim(W/\mathcal{I})^{1/2}$

   ($\mathcal{I}$ is prime)

2. $\operatorname{codim}_{\mathcal{W}} \mathcal{J}_+ = \operatorname{mult} \mathcal{J} = \operatorname{mult}_{\overline{Ge}} \operatorname{gr} \mathcal{J}$

Goal: compare $U(\mathfrak{g})$, $A_V \otimes \mathcal{W}$, $V = [\mathfrak{g}, f]$

$(e, [\cdot,\cdot])$ sympl form on V.

"$U(\mathfrak{g})$, $A_V \otimes \mathcal{W}$ are isomorphic up to completions"

Darboux Weinstein theorem.

1. $T^*G \supset G(1,e) \rightsquigarrow \widehat{T^*G}_{G(1,e)}$
2. $X \times V^* \supset G(1,e) \rightsquigarrow \widehat{(X \times V^*)}_{G(1,e)}$

$$\widehat{T^*G}_{G(1,e)} \cong \widehat{(X \times V^*)}_{G(1,e)}$$